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A new generalization of vector-coherent state theory for the SO(5) $\supset$ U(2) proton–neutron quasispin group

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Abstract. The introduction of a set of intrinsic coordinates to give an explicit construction of the intrinsic states of vector-coherent state theory has greatly simplified earlier attempts to generalize this theory to include operators lying outside the group algebra. Very explicit vector-coherent state constructions of such operators can now be given in terms of vector-coupled combinations of intrinsic and collective operators. When organized into tensors which induce specific shifts in irreducible representations these lead to the reduced Wigner coefficients needed in practical calculations. The SO(5) $\supset$ U(2) proton–neutron quasispin algebra is used as an example to give further simplifications of earlier results. All Wigner coefficients needed to give the n, T-dependence of matrix elements in the seniority scheme can now be given through a few terms expressed solely through angular momentum recoupling coefficients and the K-matrix elements of vector-coherent state theory.

1. Introduction

Vector-coherent state (VCS) theory [1–6] and its associated K-matrix technique [1, 2, 7, 8] are now well established as powerful tools for the evaluation of the matrix representations of higher-rank Lie algebras and their non-compact generalizations. VCS theory gives a very explicit method for the construction of the irreducible representations of a full group algebra from the irreducible representations of a 'core' subalgebra by an inductive process [9], in the language of quantum theory by a vector-coupling process which couples the 'intrinsic' or 'internal' (or 'spin') states with the 'collective' (or 'orbital') states whose excitations are realized in VCS theory in terms of polynomials in a set of complex Bargmann space variables, z_i. Matrix elements of the group generators then follow directly from a knowledge of the subgroup recoupling (Racah) coefficients and the matrix elements of the intrinsic components of the generators. The latter follow from a knowledge of the generator matrix elements of the core subalgebras. Like the electron-spin matrix elements, they do not require a knowledge of explicit 'spin' or 'intrinsic' or 'internal' degrees of freedom. In order to determine the full Wigner–Racah calculus of higher-rank algebras, recent interest in this field has focused on the problem of finding the VCS realizations of operators lying outside the group algebra. In a recent attempt to generalize VCS theory [10–12], coherent state realizations of such operators have also been given in terms of a set of intrinsic operators which are vector-coupled to collective z-space operators. In this method, the intrinsic operators are again defined through their actions on intrinsic states, that is through an evaluation of their matrix elements. Unlike the well known matrix elements of an 'intrinsic spin' operator or an intrinsic generator of an arbitrary core subalgebra, the matrix elements of the intrinsic components of operators lying outside the group algebra are now much more complicated for two reasons: such intrinsic operators will connect an intrinsic state of an irreducible representation, ω, to (i) intrinsic states belonging to different irreducible representations, ω'; and (ii) a set of (collective $\times$ intrinsic) states, since the action of such an intrinsic operator on an intrinsic state can now induce collective excitations as well as a change in irreducible representation. Despite these difficulties, the intrinsic components of simple operators were defined through their non-zero reduced matrix elements in a number of examples. Many of the simple Wigner coefficients were evaluated by this method for the neutron–proton quasispin group [10], SO(5) $\supset$ U(2), for the Sp(6) $\supset$ U(3) branch of the fermion dynamical symmetry group [11] and the canonical subgroup branch of the unitary group [12], U(3) $\supset$ U(2) $\times$ U(1). However, the generalized VCS method described in [10–12] is quite cumbersome.

A better method was recently proposed by LeBlanc [13] who was guided by the work of Bouwknegt *et al* [14] on two-dimensional conformal field theories. In this newest generalization of VCS theory, a set of intrinsic coordinates $q_j, j = 1, \ldots, \ell =$ rank of the full group, is introduced. These q_j are used to construct the highest (or lowest) weight components of the intrinsic state, the remaining intrinsic states being generated through a set of Bargmann variables for the core subalgebra. A very explicit construction can now be given for the intrinsic components of operators lying outside the group algebra through the intrinsic q_j, their conjugates p_j, and the subgroup Bargmann variables, z_i, and their conjugate derivative operators. The VCS realization of an arbitrary operator can then be given by vector-coupled combinations of intrinsic and collective tensor operators. These are organized into operators of irreducible tensor rank for the full group with specific shift properties, that is they induce very definite shifts in the irreducible representation when acting on a generic state of arbitrary ω. The matrix elements of these tensor operators, when suitably normalized, lead at once to the necessary reduced Wigner coefficients of the full group, where these are expressed entirely in terms of recoupling coefficients of the core subgroup and the K-matrix elements of VCS theory.

The new VCS method was applied to the standard canonical group chain U(n) $\supset$ U($n-1$)$\times$U(1) in [13] (for an introduction, for the special case $n = 3$, see [15]). Since the U(n) group chain is special, it may be instructive to apply the new generalization of VCS theory to another example. The simple rank-2 group SO(5) with its U(2) subgroup, as realized by the neutron–proton quasispin group of nuclear spectroscopy, is ideal for this purpose since its subgroup recoupling coefficients are readily available. The earlier generalization of VCS theory has already achieved the desired result for some of the simplest SO(5) $\supset$ U(2) reduced Wigner coefficients [10]. That is, these have been expressd very simply in terms of SU(2) recoupling coefficients and the K-matrix elements of SO(5). For some more challenging Wigner coefficients, however, the expressions in [10] involve some intermediate state sums. The power of the new technique can, therefore, be used to achieve further simplifications and leads to a truly viable Wigner–Racah calculus for the neutron–proton quasispin group.

In an alternate approach, A Klein has recently called attention to the connection between the quantized Bogoliubov transformation [16,17] and the problem of constructing the VCS realizations of operators lying outside the group algebra. The SO(5) neutron–proton quasispin algebra, or its isomorphic Sp(4) algebra, was used by Klein *et al* [18] to explore this connection through a boson–quasifermion mapping

[19,20] of the neutron–proton shell model algebra. It is interesting to note that the quasifermion creation and annihilation operators of this approach share some of the properties of the isospin-$\frac{1}{2}$ operators which can be used as the basic building blocks for the intrinsic operators in the new generalization of the VCS method [21].

2. The new VCS realization of the neutron–proton quasispin algebra

The generators of the SO(5) algebra can be split into

(i) a set of three commuting raising generators, the $J = 0$, $T = 1$ pair creation operators of the single j-shell,

$$A^\dagger(M_T) = \frac{1}{2}\sum_m \sum_{m_{t_1}} (-1)^{j-m} a^\dagger_{jmm_{t_1}} a^\dagger_{j-mm_{t_2}} \langle \tfrac{1}{2}m_{t_1} \tfrac{1}{2}m_{t_2}|1M_T\rangle \quad (1a)$$

where $a^\dagger_{jmm_t}$ are single-nucleon creation operators

(ii) a set of conjugate lowering operators

$$A(M_T) = (A^\dagger(M_T))^\dagger \quad (1b)$$

(iii) the generators of the SU(2)×U(1) core subalgebra made up of the isospin generators $\boldsymbol{T}$ and the number operator

$$\begin{aligned} T_{\pm 1} &= \mp\frac{1}{\sqrt{2}}\sum_m a^\dagger_{jm\pm\frac{1}{2}} a_{jm\mp\frac{1}{2}} \\ T_0 &= \frac{1}{2}\sum_m (a^\dagger_{jm+\frac{1}{2}} a_{jm+\frac{1}{2}} - a^\dagger_{jm-\frac{1}{2}} a_{jm-\frac{1}{2}}) \\ N_{\text{op}} &= \sum_m \sum_{m_t} a^\dagger_{jmm_t} a_{jmm_t} \end{aligned} \quad (1c)$$

with standard Cartan SO(5) operators

$$H_1 = \tfrac{1}{2}N_{\text{op}} - \Omega \qquad H_2 = T_0. \quad (1d)$$

A generalization to multi-j shell configurations merely requires the inclusion of a j-sum in all summations and a replacement of $\Omega = (j + \frac{1}{2})$ by the full pair-degeneracy number.

In the VCS formalism state vectors are mapped onto their z-space functional realization

$$|\psi\rangle \to \psi_\omega(z) = \langle\omega|e^{z\cdot A}|\psi\rangle \quad (2a)$$

with

$$(z \cdot A) = \sum_M z_M A_M \qquad (M \equiv M_T) \quad (2b)$$

where we have used the standard spherical tensor form for the pair annihilation operators,

$$A_{M_T} \boxminus (-1)^{1-M_T} A(-M_T). \tag{3a}$$

Note also that it is the z^*_M which transform as standard spherical tensors of rank 1, spherical component M, whereas the z_M are related to standard spherical tensors $Z_{1,M}$ via

$$Z_{1,M} = (-1)^{1-M} z_{-M} \tag{3b}$$

so that in terms of the Cartesian z_1, z_2, z_3

$$Z_{1,\pm 1} = z_{\mp 1} = \pm\frac{1}{\sqrt{2}}(z_1 \pm \mathrm{i} z_2) \qquad Z_{1,0} = -z_0 = -z_3. \tag{3c}$$

In equation (2*a*), $|\omega\rangle$, the so-called intrinsic state, is a state of an irreducible representation of the U(2) core subgroup

$$|\omega\rangle \equiv |n = v; t\ m_t\rangle \tag{4}$$

with nucleon number = seniority number, $n = v$; and with isospin = reduced isospin, given by t. Note that $|\omega\rangle$ is a $(2t+1)$-dimensional vector which is annihilated by the $J = 0$, $T = 1$ pair annihilation operators, A_{M_T}.

In VCS theory operators, O, are mapped into their z-space realizations, $\Gamma(O)$, via

$$\begin{aligned} O|\psi\rangle \to \Gamma(O)\psi_\omega(z) &= \langle\omega|\mathrm{e}^{z\cdot A}O|\psi\rangle = \langle\omega|(\mathrm{e}^{z\cdot A}O\mathrm{e}^{-z\cdot A})\mathrm{e}^{z\cdot A}|\psi\rangle \\ &= \langle\omega|\{O + [z\cdot A, O] + \tfrac{1}{2}[z\cdot A,[z\cdot A, O]] + \cdots\}\mathrm{e}^{z\cdot A}|\psi\rangle \end{aligned} \tag{5}$$

where the $\Gamma(O)$ for the generators are given in Cartesian form through equations (9) in [10]. For present purposes, it may be useful to give the needed commutators in standard spherical tensor form

$$\begin{array}{lll} [A_{+1}, T_{-1}] = -A_0 & [A_{+1}, T_0] = -A_{+1} & [A_{+1}, T_{+1}] = 0 \\ [A_0, T_{-1}] = -A_{-1} & [A_0, T_0] = 0 & [A_0, T_{+1}] = A_{+1} \\ [A_{-1}, T_{-1}] = 0 & [A_{-1}, T_0] = A_{-1} & [A_{-1}, T_{+1}] = A_0 \\ [T_{-1}, T_{-1}] = 0 & [T_{-1}, T_0] = T_{-1} & [T_{-1}, T_{+1}] = T_0 \end{array} \tag{6a}$$

and, with $A^\dagger_{M_T} \equiv A^\dagger(M_T)$,

$$\begin{array}{lll} [A_{+1}, A^\dagger_{+1}] = 0 & [A_{+1}, A^\dagger_0] = T_{+1} & [A_{+1}, A^\dagger_{-1}] = T_0 - H_1 \\ [A_0, A^\dagger_{+1}] = -T_{+1} & [A_0, A^\dagger_0] = H_1 & [A_0, A^\dagger_{-1}] = T_{-1} \\ [A_{-1}, A^\dagger_{+1}] = -T_0 - H_1 & [A_{-1}, A^\dagger_0] = -T_{-1} & [A_{-1}, A^\dagger_{-1}] = 0 \\ [T_{-1}, A^\dagger_{+1}] = A^\dagger_0 & [T_{-1}, A^\dagger_0] = A_{-1} & [T_{-1}, A^\dagger_{-1}] = 0. \end{array} \tag{6b}$$

The SO(5) basis vectors of ordinary Hilbert space are mapped into the vector-coupled coherent state basis

$$|(\omega_1\omega_2)niTM_T\rangle \rightarrow |v, p[T_p \times t]TM_T\rangle \tag{7}$$

where $(\omega_1\omega_2) = (\Omega - \frac{1}{2}v, t)$ are the SO(5) irrep labels, n is the nucleon number given by $n = v + 2p$, with p = number of $J = 0$, $T = 1$ coupled nucleon pairs which are combined with the v-nucleon configuration entirely free of such pairs. The VCS basis vectors are given by

$$|v, p[T_p \times t]TM_T\rangle = [Z^{(p0)}_{T_p}(\boldsymbol{z}) \times |v; t\rangle]^T_{M_T} \tag{8}$$

where $Z^{(p0)}(z)$ is a z-space solid harmonic of degree p (see equation (15) in [10]), with isospin $T_p = p, p-2, \ldots, 0$ (or 1), the isospin of the p symmetrically coupled $J = 0$, $T = 1$ pairs. (Note that $Z^{(10)}_{1,m} \equiv Z_{1,m}$.) In equation (8), the square bracket denotes the vector-coupling (in a right-to-left coupling order convention) of the intrinsic t with the collective T_p to resultant total isospin T. The label, i, in equation (7) stands for the fourth quantum number of SO(5). In VCS theory, it is given naturally through the unitarization K-matrix; see, in particular, equations (22)–(24) and appendix A in [10]. This K-matrix also converts the reduced matrix element of an arbitrary operator of spherical tensor rank τ to the VCS matrix elements in the simple vector-coupled basis of equation (8) through

$$\begin{aligned}\langle(\omega_1'\omega_2')n'i'T'\|O^\tau\|(\omega_1\omega_2)niT\rangle \\ = \sum_{T_p',T_p} (K^{-1})_{i'T_p'}\langle v', p'[T_p' \times t']T'\|\Gamma(O)^\tau\|v, p[T_p \times t]T\rangle(K)_{T_p i}.\end{aligned} \tag{9}$$

In the new generalization of VCS theory, a seemingly backward step is made first. The core subalgebra of VCS theory is replaced by the simpler Cartan subalgebra H_1, H_2; and the $(2t+1)$-dimensional intrinsic state $|v; tm_t\rangle$ is replaced by the one-dimensional lowest-weight (LW) state $|v; tm_t = -t\rangle$. The VCS factor $(\boldsymbol{z} \cdot \boldsymbol{A})$ must then also be replaced by a new VCS factor involving all the lowering operators of the full Cartan lowering type with a new set of (primed) VCS variables $z'_{-\alpha} \equiv (z'_{+1}, z'_0, z'_{-1}, \zeta')$;

$$\boldsymbol{z} \cdot \boldsymbol{A} = \sum_{M=-1}^{+1} z_M A_M \rightarrow \boldsymbol{z}' \cdot \boldsymbol{E} = \sum_\alpha z'_{-\alpha} E_{-\alpha} = \sum_{M=-1}^{+1} (z'_M A_M) + \zeta' T_{-1} \tag{10}$$

where the lowering operators $E_{-\alpha}$, with roots $-\alpha = -(e_1 - e_2), -e_1, -(e_1 + e_2)$, and $-e_2$, are A_{+1}, A_0, A_{-1}, and T_{-1}, respectively. Note that the primed VCS variable associated with the SU(2) subgroup operator T_{-1} is named ζ'; i.e. $z'_{-e_2} \equiv \zeta'$. This apparent retrogression in the new approach leads to a more complicated form of the VCS realizations $\Gamma'(O)$ for the generators since the lowering operator T_{-1} does not commute with the full set of lowering operators, A_M. It is useful to write $e^{\boldsymbol{z}' \cdot \boldsymbol{E}}$ in both a right and left subgroup form by making repeated use of the Campbell–Baker–Hausdorff relation

$$e^A e^B = e^{\{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\cdots\}} \tag{11}$$

to yield

$$\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} = \mathrm{e}^{z'_{+1}A_{+1}}\mathrm{e}^{(z'_0+\frac{1}{2}z'_{+1}\zeta')A_0}\mathrm{e}^{(z'_{-1}+\frac{1}{2}z'_0\zeta'+\frac{1}{6}z'_{+1}\zeta'^2)A_{-1}}\mathrm{e}^{\zeta' T_{-1}} \tag{12}$$

and

$$\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} = \mathrm{e}^{\zeta' T_{-1}}\mathrm{e}^{z'_{+1}A_{+1}}\mathrm{e}^{(z'_0-\frac{1}{2}z'_{+1}\zeta')A_0}\mathrm{e}^{(z'_{-1}-\frac{1}{2}z'_0\zeta'+\frac{1}{6}z'_{+1}\zeta'^2)A_{-1}}. \tag{13}$$

These lead to

$$\begin{aligned}
\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}A_{-1} &= \frac{\partial}{\partial z'_{-1}}\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}A_0 &= \left(\frac{\partial}{\partial z'_0} + \frac{1}{2}\zeta'\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}A_{+1} &= \left(\frac{\partial}{\partial z'_{+1}} + \frac{1}{2}\zeta'\frac{\partial}{\partial z'_0} + \frac{1}{12}\zeta'^2\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}T_{-1} &= \left(\frac{\partial}{\partial \zeta'} - \frac{1}{2}z'_{+1}\frac{\partial}{\partial z'_0} - \frac{1}{2}z'_0\frac{\partial}{\partial z'_{-1}} - \frac{1}{12}z'_{+1}\zeta'\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
A_{-1}\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} &= \frac{\partial}{\partial z'_{-1}}\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
A_0\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} &= \left(\frac{\partial}{\partial z'_0} - \frac{1}{2}\zeta\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
A_{+1}\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} &= \left(\frac{\partial}{\partial z'_{+1}} - \frac{1}{2}\zeta'\frac{\partial}{\partial z'_0} + \frac{1}{12}\zeta'^2\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} \\
T_{-1}\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}} &= \left(\frac{\partial}{\partial \zeta'} + \frac{1}{2}z'_{+1}\frac{\partial}{\partial z'_0} + \frac{1}{2}z'_0\frac{\partial}{\partial z'_{-1}} - \frac{1}{12}z'_{+1}\zeta'\frac{\partial}{\partial z'_{-1}}\right)\mathrm{e}^{\boldsymbol{z}'\cdot\boldsymbol{E}}.
\end{aligned} \tag{15}$$

Equations (14) lead at once to the VCS realizations of the lowering operators. For example

$$\Gamma'(A_0) = \frac{\partial}{\partial z'_0} + \frac{1}{2}\zeta'\frac{\partial}{\partial z'_{-1}}. \tag{16}$$

The remaining generators follow from the commutator expansion of equation (5) including triple and quadruple commutators, together with equations (15); leading, e.g., to

$$\begin{aligned}
\Gamma'(T_{+1}) = {} & -t\zeta' + \frac{1}{2}\zeta'^2\frac{\partial}{\partial\zeta'} + \left(z'_0 - \frac{1}{2}\zeta' z'_{+1}\right)\frac{\partial}{\partial z'_{+1}} \\
& + \left(z'_{-1} + \frac{1}{6}\zeta'^2 z'_{+1}\right)\frac{\partial}{\partial z'_0} + \left(\frac{1}{2}\zeta' z'_{-1} + \frac{1}{12}\zeta'^2 z'_0\right)\frac{\partial}{\partial z'_{-1}}.
\end{aligned} \tag{17}$$

Note the entanglement between the collective z'_M and the subgroup operator, ζ'. The collective and intrinsic variables can be disentangled to lead to the original VCS realization for the generators, see equations (9) in [10], via the nonlinear transformation to new z_M, ζ,

$$\begin{aligned} &z_{+1} = z'_{+1} \qquad z_0 = z'_0 - \tfrac{1}{2}\zeta' z'_{+1} \\ &z_{-1} = z'_{-1} - \tfrac{1}{2}\zeta' z'_0 + \tfrac{1}{6}\zeta'^2 z'_{+1} \qquad \zeta = \zeta'. \end{aligned} \tag{18}$$

In addition, it is possible to introduce a set of intrinsic coordinates q_1, q_2, with a set of canonically conjugate p_1, p_2,

$$[p_j, q_k] = -\mathrm{i}\delta_{jk}. \tag{19}$$

The lowest-weight (LW) state can then be written explicitly in terms of the q_j

$$|LW\rangle = |v; t, m_t = -t\rangle = \mathrm{e}^{-\mathrm{i}\omega_1 q_1 - \mathrm{i}\omega_2 q_2}|0\rangle \tag{20}$$

with $\omega_2 = t$, and with an intrinsic space inner product defined such that $\langle LW|LW\rangle = 1$. The intrinsic q_j, p_j can therefore be viewed as internal angle-action variables defined over the angle interval 0 to 1. Note also that $p_j|LW\rangle = -\omega_j|LW\rangle$.

The intrinsic state construction of the full $(2t+1)$-dimensional intrinsic space now proceeds via the SU(2) coherent state construction

$$|v; t, m_t = -t + k\rangle = \frac{\zeta^k}{\sqrt{k!}}|LW\rangle \tag{21a}$$

with an SU(2) unitarization K-operator which collapses to a simple one-dimensional normalization factor

$$K_k = \sqrt{\frac{(2t)!}{(2t-k)!2^k}} \tag{21b}$$

with $k = 0, \ldots, 2t$ for $m_t = -t, \ldots, +t$.

With the shorthand notation (in the new variables)

$$\frac{\partial}{\partial z_M} \equiv \partial_M \qquad \frac{\partial}{\partial \zeta} \equiv \partial_\zeta \tag{22}$$

the VCS realization of the generators can be put in the form

$$\begin{aligned} &\Gamma(A_M) = \partial_M \equiv \partial_{1,M} \\ &\Gamma(T_M) = T_M + \sqrt{2}[Z_1 \times \partial_1]^1_M = T_M^{\mathrm{intr.}} + T_M^{\mathrm{coll.}} \\ &\Gamma(H_1) = \mathcal{H}_1 + (z \cdot \partial) = \mathcal{H}_1^{\mathrm{intr.}} + H_1^{\mathrm{coll.}} \\ &\Gamma(A_M^\dagger) = -Z_{1,M}\mathcal{H}_1 + \sqrt{2}[Z_1 \times T_1^{\mathrm{intr.}}]^1_M - \tfrac{1}{2}(z \cdot z)\partial_{1,M} - Z_{1,M}(z \cdot \partial) \end{aligned} \tag{23}$$

which is the spherical tensor analogue of the Cartesian form in [10]. (Note that the $\partial_{1,M}$ are standard spherical tensors of rank 1, spherical component M;

whereas the z_M are related to the spherical tensors $Z_{1,M}$ via equations (3); so that $(z \cdot \partial) = +\sqrt{3}[Z_1 \times \partial_1]^0_0$ whereas $(z \cdot z) = -\sqrt{3}[Z_1 \times Z_1]^0_0$. The square bracket will denote vector-coupling in a right-to-left coupling order convention throughout.) The U(2) subgroup generators are written as sums of purely intrinsic and purely collective operators. Now, however, the intrinsic components of these generators can be given in very specific form in terms of a set of intrinsic operators (to be written in calligraphic letters)

$$\begin{aligned} &\mathcal{H}_1 = p_1 \\ &\mathcal{T}_{-1} = \partial_\zeta \qquad \mathcal{T}_0 = p_2 + \zeta\partial_\zeta \qquad \mathcal{T}_{+1} = \zeta p_2 + \tfrac{1}{2}\zeta^2\partial_\zeta . \end{aligned} \tag{24}$$

All operators, including those lying outside the group algebra, can now be defined in terms of a set of intrinsic operators constructed from the intrinsic

$$q_1, q_2 \qquad p_1, p_2 \qquad \zeta, \partial_\zeta$$

and a set of collective operators constructed through the collective

$$z_{+1}, z_0, z_{-1} \qquad \partial_{+1}, \partial_0, \partial_{-1}.$$

Moreover, if these are expressed in terms of vector-coupled combinations of the form

$$[(T^{\text{coll.}}(z, \partial))_{\tau_1} \times (T^{\text{intr.}}(q_i, p_i, \zeta, \partial_\zeta))_{\tau_2}]^\tau_{m_\tau}$$

their matrix elements can be expressed in terms of standard recoupling coefficients in the vector-coupled basis of equation (8).

3. Irreducible shift tensors

The aim of the new VCS method is to organize operators outside the group algebra not only into sets of irreducible tensor operators with definite weights but into sets of irreducible tensor operators which induce very definite shifts in the irreducible representation when acting on a generic state of arbitrary SO(5) irrep $(\omega_1, \omega_2) = (\Omega - \frac{1}{2}v, t)$. The single-nucleon creation and annihilation operators serve as a simplest example. With fixed jm, the four operators $a^\dagger_{jmm_t=\pm\frac{1}{2}}; \pm(-1)^{j-m}a_{j,-m,m_t=\mp\frac{1}{2}}$, span the four-dimensional irrep $(\frac{1}{2}\frac{1}{2})$ of SO(5), with weight points $+\frac{1}{2}, \pm\frac{1}{2}$; and $-\frac{1}{2} \pm \frac{1}{2}$, respectively. That is, these operators shift the weights H_1, H_2 by $+\frac{1}{2}, \pm\frac{1}{2}$; or $-\frac{1}{2} \pm \frac{1}{2}$ when acting on generic states of arbitrary weights. Each one of these will induce shifts in the irrep $(\omega_1, \omega_2) \rightarrow (\omega_1', \omega_2') = (\omega_1 + \Delta_1, \omega_2 + \Delta_2)$, with $\Delta_1\Delta_2 = +\frac{1}{2} + \frac{1}{2},\ +\frac{1}{2} - \frac{1}{2},\ -\frac{1}{2} + \frac{1}{2},\ -\frac{1}{2} - \frac{1}{2}$ when acting on generic states of (ω_1, ω_2); i.e. the shifts range over the same set of numbers as the weights. A nucleon creation (or annihilation) operator will, in general, induce all four of these shifts, whereas an irreducible shift tensor is to select one specific shift, $\Delta_1\Delta_2$, out of the four possibilities. Operators will thus be labelled by their irreducible tensor rank, $(\lambda_1\lambda_2)$, by their shift, $\Delta_1\Delta_2$, and by their subgroup labels, $h_1; \tau h_2 \equiv m_\tau$, so that

$$\begin{aligned} &T(\lambda_1\lambda_2)^{\Delta_1\Delta_2}_{h_1;\tau m_\tau}|(\omega_1, \omega_2)H_1 i T M_T\rangle \\ &\qquad \rightarrow |(\omega_1 + \Delta_1, \omega_2 + \Delta_2)(H_1 + h_1)i'T'(M_T + m_\tau)\rangle . \end{aligned} \tag{25}$$

It will be assumed that the irreps $(\lambda_1\lambda_2)$ are simple enough so that the SU(2)×U(1) subgroup labels $h_1; \tau m_\tau$ are sufficient to label the tensors. This would be true, for example, for the irreps $(\frac{1}{2}\frac{1}{2}),(\frac{3}{2}\frac{1}{2}),(10),(11),(20),(21),(22)$, of greatest interest in nuclear spectroscopy. Irreps, such as $(\frac{3}{2}\frac{1}{2}),(11),(20),(21),(22)$, with multiple weight points will lead to multiple solutions for the shift tensors of the corresponding $\Delta_1\Delta_2$, which will require additional upper-index shift labels.

To construct the shift tensors of definite $\Delta_1\Delta_2$, it will be usful to introduce a set of 'screening charges', (the language comes from the field theory applications in [14]). For these we need the left VCS realizations for the Cartan lowering operators, $\Gamma_{\text{left}}(E_{-\alpha})$. Recall that

$$\Gamma(E_{-\alpha_o})\psi(z') = \langle LW|\mathrm{e}^{z'\cdot E}E_{-\alpha_o}|\psi\rangle. \tag{26}$$

Now, define $\Gamma_{\text{left}}(E_{-\alpha_o})$ by

$$\Gamma_{\text{left}}(E_{-\alpha_o})\psi(z') = \langle LW|(-E_{-\alpha_o})\mathrm{e}^{z'\cdot E}|\psi\rangle \tag{27}$$

After transformation from the $z'_{-\alpha} \equiv z'_M, \zeta'$ to z_M, ζ via equations (18), these are

$$\begin{aligned}
&\Gamma_{\text{left}}(A_{-1}) = -\partial_{-1} \qquad \Gamma_{\text{left}}(T_{-1}) = -\partial_\zeta, \\
&\Gamma_{\text{left}}(A_0) = -(\partial_0 - \zeta\partial_{-1}) \\
&\Gamma_{\text{left}}(A_{+1}) = -(\partial_{+1} - \zeta\partial_0 + \tfrac{1}{2}\zeta^2\partial_{-1}).
\end{aligned} \tag{28}$$

Note that the minus sign in the defining equation (27) is needed to preserve the generator commutation relations among the $\Gamma_{\text{left}}(E_{-\alpha})$. The screening charge for the root α is now defined by

$$S_\alpha = \mathrm{e}^{\mathrm{i}(\alpha\cdot q)}\Gamma_{\text{left}}(E_{-\alpha}) \tag{29a}$$

that is

$$\begin{aligned}
&S_{e_1-e_2} = -\mathrm{e}^{\mathrm{i}(q_1-q_2)}(\partial_{+1} - \zeta\partial_0 + \tfrac{1}{2}\zeta^2\partial_{-1}) \\
&S_{e_1} = -\mathrm{e}^{\mathrm{i}q_1}(\partial_0 - \zeta\partial_{-1}) \\
&S_{e_1+e_2} = -\mathrm{e}^{\mathrm{i}(q_1+q_2)}\partial_{-1} \qquad S_{e_2} = -\mathrm{e}^{\mathrm{i}q_2}\partial_\zeta.
\end{aligned} \tag{29b}$$

Note, that the commutator $[S_{\alpha_1}, S_{\alpha_2}] = 0$ if $\alpha_1 + \alpha_2$ is not a root, and that

$$[S_{e_2}, S_{e_1-e_2}] = S_{e_1} \qquad [S_{e_2}, S_{e_1}] = S_{e_1+e_2}. \tag{30}$$

The screening charges have the following properties. The screening charge for $\alpha = \nu_1 e_1 + \nu_2 e_2$ in its left action on an intrinsic state $(\omega_1\omega_2)$ not only shifts the intrinsic state to $(\omega_1 + \nu_1, \omega_2 + \nu_2)$ but also leaves the subgroup labels $H_1 T M_T$ invariant since S_α is a U(2) subgroup scalar. The construction of the irreducible shift tensors $T(\lambda_1\lambda_2)^{\Delta_1\Delta_2}_{h_1,\tau m_\tau}$ now proceeds in three steps.

Step 1. Construction of the maximal-weight, maximal-shift tensor.

Step 2. Construction of the maximal-weight, lesser-shift tensors through the use of screening charges.

Step 3. Construction of arbitrary-weight, specific-shift tensors by repeated application of commutator-taking with the raising generators $\Gamma(A_M^\dagger)$.

In step 1, it is most natural to choose the LW, as the maximal-weight, since our irreps are induced from states with the minimum number of particles, $n = v$, and $m_t = -t$. The maximal-shift LW tensor follows from our state construction in terms of specific intrinsic q_1, q_2,

$$T(\lambda_1\lambda_2)_{LW}^{\lambda_1\lambda_2} = \mathrm{e}^{-\mathrm{i}\lambda_1 q_1 - \mathrm{i}\lambda_2 q_2}. \tag{31}$$

The LW tensors of arbitrary shift, $\Delta_1\Delta_2$, are then built through the screening charges from linear combinations of the form

$$\mathrm{e}^{-\mathrm{i}\lambda_1 q_1 - \mathrm{i}\lambda_2 q_2} \prod_i S_{\alpha^{(i)}} \tag{32}$$

where, with $\alpha^{(i)} = \nu_i^{(i)} e_1 + \nu_2^{(i)} e_2$, $\Delta_1 = \lambda_1 - \sum_i \nu_1^{(i)}, \Delta_2 = \lambda_2 - \sum_i \nu_2^{(i)}$.

The basic relation which is used to construct the LW lesser-shift tensors is given by [13, 14]

$$S_\alpha^{[(2(\omega'\cdot\alpha)/(\alpha\cdot\alpha))+1]} T(\lambda_1\lambda_2)_{LW}^{\Delta_1\Delta_2} = T(\lambda_1\lambda_2)_{LW}^{\Delta_1'\Delta_2'} S_\alpha^{[(2(\omega\cdot\alpha)/(\alpha\cdot\alpha))+1]} \tag{33}$$

where α is one of the simple roots, $e_1 - e_2$ or e_2; $(\omega_1'\omega_2') = (\omega_1 + \Delta_1, \omega_2 + \Delta_2)$; and $\Delta_1'\Delta_2'$ is obtained from $\Delta_1\Delta_2$ by a Weyl reflection in the plane normal to α; i.e. for $\alpha = e_2 :\ \Delta_1'\Delta_2' = \Delta_1, -\Delta_2$; while for $\alpha = e_1 - e_2 :\ \Delta_1'\Delta_2' = \Delta_2\Delta_1$.

Step 3 in the shift-operator construction procedure is achieved by repeated application of the commutator relation

$$\begin{aligned}&[\Gamma(A_M^\dagger), T(\lambda_1\lambda_2)_{h_1;\tau m_\tau}^{\Delta_1\Delta_2}] \\ &\quad = \sum_{\tau'} T(\lambda_1\lambda_2)_{h_1+1;\tau' m_\tau + M}^{\Delta_1\Delta_2} \langle(\lambda_1\lambda_2)h_1{+}1, \tau' m_\tau{+}M|A_M^\dagger|(\lambda_1\lambda_2)h_1\tau m_\tau\rangle\end{aligned} \tag{34}$$

where the matrix elements of the generators are known from the VCS construction. It should perhaps be emphasized that these $T(\lambda_1\lambda_2)_{h_1;\tau m_\tau}^{\Delta_1\Delta_2}$ are the VCS realizations of the shift tensors; i.e. they are $\Gamma(T)$. For simplicity of notation, we shall dispense with the symbol Γ.

4. The fundamental SO(5) spinors

The shift-tensor construction process will be illustrated first with the simplest irreducible tensors, $T(\frac{1}{2}\frac{1}{2})$, spanned by the single-nucleon creation and annihilation operators with $m_t = \pm\frac{1}{2}$. The LW shift tensor with $\Delta_1\Delta_2 = +\frac{1}{2}, +\frac{1}{2}$ is given by equation (31). The LW shift tensor with $\Delta_1\Delta_2 = +\frac{1}{2}, -\frac{1}{2}$ is obtained from this maximal shift tensor by a simple Weyl reflection in the plane normal to e_2 by an application of equation (33) with $\alpha = e_2$. A second reflection, now in the plane normal to $e_1 - e_2$, gives the LW shift tensor with $\Delta_1\Delta_2 = -\frac{1}{2}, +\frac{1}{2}$, while a final reflection in the plane normal to e_2 gives the LW tensor with $\Delta_1\Delta_2 = -\frac{1}{2}, -\frac{1}{2}$, where the application of equation (33) requires repeated use of the commutator relations of equation (30). The results are

$$
\begin{aligned}
T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},+\frac{1}{2}}_{LW} &= \mathrm{e}^{-\mathrm{i}q_1/2-\mathrm{i}q_2/2} \\
T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},-\frac{1}{2}}_{LW} &= \mathrm{e}^{-\mathrm{i}q_1/2+\mathrm{i}q_2/2}\partial_\zeta \\
T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},+\frac{1}{2}}_{LW} &= \mathrm{e}^{+\mathrm{i}q_1/2-\mathrm{i}q_2/2}\{(\partial_{+1} - \zeta\partial_0 + \tfrac{1}{2}\zeta^2\partial_{-1})\partial_\zeta + (\partial_0 - \zeta\partial_{-1})(p_1 - p_2 - 1)\} \\
T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},-\frac{1}{2}}_{LW} &= \mathrm{e}^{+\mathrm{i}q_1/2+\mathrm{i}q_2/2}\{(\partial_{+1} - \zeta\partial_0 + \tfrac{1}{2}\zeta^2\partial_{-1})\partial_\zeta^2 \\
&\quad + (\partial_0 - \zeta\partial_{-1})\partial_\zeta(p_1 - p_2 - 2) - \partial_{-1}p_2(2p_1 - 3)\}.
\end{aligned} \tag{35}
$$

The simplest shift tensors are those with $\Delta_1 = +\frac{1}{2}$. For these the full set of tensors, (all weights as obtained through equation (34)), can be expressed in terms of two intrinsic spinors $\mathcal{A}$ and $\mathcal{B}$

$$
T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},+\frac{1}{2}}_{-\frac{1}{2};\frac{1}{2}m_t} = \mathrm{e}^{-\mathrm{i}q_1/2}\mathcal{A}_{m_t} \qquad T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},+\frac{1}{2}}_{+\frac{1}{2};\frac{1}{2}m_t} = \mathrm{e}^{-\mathrm{i}q_1/2}\sqrt{\tfrac{3}{2}}[Z_1 \times \mathcal{A}_{\frac{1}{2}}]^{\frac{1}{2}}_{m_t} \tag{36}
$$

$$
T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},-\frac{1}{2}}_{-\frac{1}{2};\frac{1}{2}m_t} = \mathrm{e}^{-\mathrm{i}q_1/2}\mathcal{B}_{m_t} \qquad T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},-\frac{1}{2}}_{+\frac{1}{2};\frac{1}{2}m_t} = \mathrm{e}^{-\mathrm{i}q_1/2}\sqrt{\tfrac{3}{2}}[Z_1 \times \mathcal{B}_{\frac{1}{2}}]^{\frac{1}{2}}_{m_t} \tag{37}
$$

where the purely intrinsic spin-$\frac{1}{2}$ operators $\mathcal{A}$ and $\mathcal{B}$ are given by

$$
\mathcal{A}_{+\frac{1}{2}} = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}q_2/2}\zeta \qquad \mathcal{A}_{-\frac{1}{2}} = \mathrm{e}^{-\mathrm{i}q_2/2} \tag{38}
$$

$$
\mathcal{B}_{+\frac{1}{2}} = \sqrt{2}\mathrm{e}^{+\mathrm{i}q_2/2}(p_2 + \tfrac{1}{2}\zeta\partial_\zeta) \qquad \mathcal{B}_{-\frac{1}{2}} = \mathrm{e}^{+\mathrm{i}q_2/2}\partial_\zeta. \tag{39}
$$

We note also that

$$
[\mathcal{A}_{\frac{1}{2}} \times \mathcal{B}_{\frac{1}{2}}]^1_M = [\mathcal{B}_{\frac{1}{2}} \times \mathcal{A}_{\frac{1}{2}}]^1_M = \mathcal{T}_M \tag{40}
$$

that is the intrinsic isospin operator $\mathcal{T}$ is obtained through a vector-coupling of the two basic intrinsic isospin-$\frac{1}{2}$ tensors. The shift operators in equations (36) and (37) have been put in a form which has achieved our basic aim. They are expressed through vector-coupled *simple* collective and intrinsic operators. In this case the collective operator is $Z_{1,M}$ and the intrinsic operators are given as very simple functions of the intrinsic operators $q_1, q_2, p_2, \zeta, \partial_\zeta$. The full matrix element of a shift operator is thus reduced to an exercise in vector coupling. For example

$$
\begin{aligned}
&\langle(\omega_1 + \tfrac{1}{2},\ t + \tfrac{1}{2})p + 1[T_p' \times t + \tfrac{1}{2}]T' \| T(\tfrac{1}{2}\tfrac{1}{2})^{+\frac{1}{2},+\frac{1}{2}}_{+\frac{1}{2};\frac{1}{2}} \| (\omega_1 t)p[T_p \times t]T\rangle \\
&\quad = \sqrt{\tfrac{3}{2}}\begin{bmatrix} t & T_p & T \\ \frac{1}{2} & 1 & \frac{1}{2} \\ (t+\frac{1}{2}) & T_p' & T' \end{bmatrix}\langle(p+1)T_p'\|Z_1\|pT_p\rangle\langle(\omega_1+\tfrac{1}{2},\ t+\tfrac{1}{2})\|\mathcal{A}_{\frac{1}{2}}\|(\omega_1 t)\rangle
\end{aligned} \tag{41}
$$

with a similar relation for the $\Delta_1\Delta_2 = +\frac{1}{2}, -\frac{1}{2}$, shift tensor. The reduced matrix elements of the intrinsic $\mathcal{A}$ and $\mathcal{B}$ follow at once from the definition of the intrinsic operators and the explicit construction of the intrinsic states through equations (21*a*, *b*). These lead to

$$\begin{aligned} \langle(\omega_1 + \tfrac{1}{2}, t + \tfrac{1}{2})||\mathcal{A}_{\frac{1}{2}}||(\omega_1 t)\rangle &= 1 \\ \langle(\omega_1 + \tfrac{1}{2}, t - \tfrac{1}{2})||\mathcal{B}_{\frac{1}{2}}||(\omega_1 t)\rangle &= \sqrt{t(2t+1)}. \end{aligned} \tag{42}$$

In equations (41), the [ ]-coefficient is the unitary form of the standard $9j$ coefficient. The reduced matrix element of the collective Z_1 is given through an SU(3) $\supset$ SO(3) Wigner coefficient by equation (19) in [10]; see also table 2 in [10]. When properly normalized, the reduced matrix elements of the shift tensors lead at once to the SO(5) $\supset$ U(2) reduced Wigner coefficients. If these normalized or *unit* shift tensors are designated with the subscript, u, they must satisfy

$$\sum_{h_1\tau}\sum_{H_1 i T} \langle(\omega_1 + \Delta_1, \omega_2 + \Delta_2)LW||T_{\mathrm{u}}(\lambda_1\lambda_2)^{\Delta_1\Delta_2}_{h_1;\tau}||(\omega_1\omega_2)H_1 i T\rangle^2 = 1. \tag{43}$$

Here, the reduced matrix element is to be taken in ordinary Hilbert space, so that equation (9) must be used together with a relation such as equation (41). For the $T(\frac{1}{2}\frac{1}{2})^{\Delta_1\Delta_2}$ tensors with $\Delta_1 = +\frac{1}{2}$, the purely intrinsic states of the left-hand side of equation (43) can be connected only to purely intrinsic states on the right-hand side via the purely intrinsic operators with $h_1 = -\frac{1}{2}$ in equations (36), (37). In this case, therefore, the T_{u} are obtained from the T by division by the intrinsic reduced matrix elements of equations (42). This then leads to the SO(5) $\supset$ U(2) Wigner coefficients in the form given in table 4 in [10].

The more complicated shift tensors with $\Delta_1 = -\frac{1}{2}$ can also be expressed in terms of vector-coupled collective and intrinsic operators, where the intrinsic operators now include pieces built by vector-coupling $\mathcal{A}$ or $\mathcal{B}$ with $\mathcal{T}$. In this form the shift tensors with LW components are

$$T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},+\frac{1}{2}}_{LW} = \mathrm{e}^{+\mathrm{i}q_1/2}\{\sqrt{2}[\partial_1 \times [\mathcal{A}_{\frac{1}{2}} \times \mathcal{T}_1]^{\frac{3}{2}}]^{\frac{1}{2}}_{-\frac{1}{2}} - \sqrt{3}[\partial_1 \times \mathcal{A}_{\frac{1}{2}}]^{\frac{1}{2}}_{-\frac{1}{2}}(p_1 - \tfrac{1}{3}p_2 - 1)\} \tag{44}$$

$$T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},-\frac{1}{2}}_{LW} = \mathrm{e}^{+\mathrm{i}q_1/2}\{\sqrt{2}[\partial_1 \times [\mathcal{B}_{\frac{1}{2}} \times \mathcal{T}_1]^{\frac{3}{2}}]^{\frac{1}{2}}_{-\frac{1}{2}} - \sqrt{3}[\partial_1 \times \mathcal{B}_{\frac{1}{2}}]^{\frac{1}{2}}_{-\frac{1}{2}}(p_1 + \tfrac{1}{3}p_2 - \tfrac{4}{3})\}. \tag{45}$$

The matrix elements now lead to two terms of the form of equation (41). (The necessary reduced matrix elements of ∂_1 and intrinsic operators such as $[\mathcal{A} \times \mathcal{T}]^{\frac{3}{2}}$ or $[\mathcal{B} \times \mathcal{T}]^{\frac{3}{2}}$ are given in section 5.) Some simplification can be achieved by a generalization of the Philadelphia–Toronto trick [21] whereby these can be reduced to matrix elements of operators $[\partial_1 \times \mathcal{A}_{\frac{1}{2}}]^{\frac{1}{2}}$ or $[\partial_1 \times \mathcal{B}_{\frac{1}{2}}]^{\frac{1}{2}}$. For example

$$T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},+\frac{1}{2}}_{LW} = [\Lambda'_{\mathrm{op}},\ \sqrt{3}[\partial_1 \times \mathcal{A}_{\frac{1}{2}}]^{\frac{1}{2}}_{-\frac{1}{2}}] \tag{46}$$

where

$$\Lambda'_{\rm op} = -(\mathcal{T}^{\rm intr.}\cdot\mathcal{T}^{\rm coll.}) - \tfrac{1}{4}(\mathcal{T}^{\rm coll.}\cdot\mathcal{T}^{\rm coll.}) - (\mathcal{T}^{\rm intr.}\cdot\mathcal{T}^{\rm intr.}) + (z\cdot\partial)(p_1 - \tfrac{3}{2} + \tfrac{1}{4}(z\cdot\partial)) \tag{47}$$

is an operator which is a subgroup scalar with simple eigenvalues. For the higher-weight components this process becomes more of a challenge since these operators consist of an even larger number of terms. For example

$$\begin{aligned} T(\tfrac{1}{2}\tfrac{1}{2})^{-\frac{1}{2},+\frac{1}{2}}_{+\frac{1}{2};\frac{1}{2}m_\tau} &= \sqrt{2}(p_1-1)(p_1-p_2)\mathcal{A}_{\frac{1}{2},m_\tau} \\ &\quad - \tfrac{1}{\sqrt{2}}\left[[Z_1\times\partial_1]^1\times[\mathcal{A}_{\frac{1}{2}}\times\mathcal{T}_1]^{\frac{3}{2}}\right]^{\frac{1}{2}}_{m_\tau} + \sqrt{\tfrac{5}{2}}\left[[Z_1\times\partial_1]^2\times[\mathcal{A}_{\frac{1}{2}}\times\mathcal{T}_1]^{\frac{3}{2}}\right]^{\frac{1}{2}}_{m_\tau} \\ &\quad + \left\{\sqrt{\tfrac{3}{2}}\left[[Z_1\times\partial_1]^0\times\mathcal{A}_{\frac{1}{2}}\right]^{\frac{1}{2}}_{m_\tau} - \sqrt{3}\left[[Z_1\times\partial_1]^1\times\mathcal{A}_{\frac{1}{2}}\right]^{\frac{1}{2}}_{m_\tau}\right\}(p_1 - \tfrac{1}{3}p_2 - 1) \end{aligned} \tag{48}$$

leading to matrix elements with five terms of the form of equation (41). Such complicated terms can be avoided by an even simpler trick, however. Reduced matrix elements of unit shift tensors with $\Delta_1 = -\frac{1}{2}$ follow from those with $\Delta_1 = +\frac{1}{2}$ by a $1\leftrightarrow 3$ interchange symmetry property of the SO(5) $\supset$ U(2) reduced Wigner coefficient. This symmetry property was used to give the expressions in table 4 in [10] for the shifts with $\Delta_1 = -\frac{1}{2}$.

5. The (10) and (11) tensors

Irreducible tensor operators transforming according to the five-dimensional irreducible representation, (10), are needed to evaluate the matrix elements of pair creation and annihilation operators and multipole operators coupled to odd J-values (see table 1 in [10]). The LW shift tensors with $\Delta_1\Delta_2 = 0,+1,0,-1$, and $-1,0$ follow from the maximal shift tensor with $\Delta_1\Delta_2 = +1,0$ by Weyl reflections in the planes normal to $e_1 - e_2, e_2$, and again $e_1 - e_2$ by an application of equation (33), while the fifth shift tensor with $\Delta_1\Delta_2 = 0,0$ is based on equation (32). The results are

$$T(10)^{+1,0}_{LW} = \mathrm{e}^{-iq_1} \tag{49}$$

$$T(10)^{0,+1}_{LW} = (\partial\cdot[\mathcal{A}_{\frac{1}{2}}\times\mathcal{A}_{\frac{1}{2}}]^1) \tag{50}$$

$$T(10)^{0,0}_{LW} = (\partial\cdot\mathcal{T}) \tag{51}$$

$$T(10)^{0,-1}_{LW} = (\partial\cdot[\mathcal{B}_{\frac{1}{2}}\times\mathcal{B}_{\frac{1}{2}}]^1) \tag{52}$$

$$T(10)^{-1,0}_{LW} = e^{+iq_1}\left\{\sum_m(-1)^m[\partial_1\times\partial_1]^2_{-m}[\mathcal{T}_1\times\mathcal{T}_1]^2_m - (\partial\cdot\partial)((p_1-1)^2 - \tfrac{1}{3}p_2(p_2-1))\right\} \tag{53}$$

The fifth shift tensor with $\Delta_1\Delta_2 = 0,0$ has been built by a linear combination of the form (see equation (32))

$$e^{-iq_1}(S_{e_1-e_2}S_{e_2} + S_{e_1}\phi(p_1, p_2))$$

where the operator function, $\phi(p_1, p_2))$, follows from equation (33). With $\alpha = e_1 - e_2$ and e_2 this leads to the relations

$$\{S_{e_1}S_{e_1-e_2}^{\omega_1-\omega_2+1}(\phi - (\omega_1 - \omega_2 + 1)) - S_{e_1}\phi S_{e_1-e_2}^{\omega_1-\omega_2+1}\}|LW\rangle = 0 \tag{54a}$$

$$\{S_{e_1}S_{e_2}^{2\omega_2+1}(2\omega_2 + 1 + \phi) + S_{e_1+e_2}S_{e_2}^{2\omega_2}(2\omega_2 + 1)(\omega_2 + \phi) - S_{e_1}\phi S_{e_2}^{2\omega_2+1}\}|LW\rangle = 0 \tag{54b}$$

where we can use $\phi(p_1, p_2)S_{e_1-e_2}^{\omega_1-\omega_2+1} = S_{e_1-e_2}^{\omega_1-\omega_2+1}\phi(p_1+\omega_1-\omega_2+1, p_2-\omega_1+\omega_2-1)$ and $\phi(p_1, p_2)S_{e_2}^{2\omega_2+1} = S_{e_2}^{2\omega_2+1}\phi(p_1, p_2 + 2\omega_2 + 1)$, so that equations (54*a*, *b*) are both satisfied by

$$\phi(p_1, p_2) = p_2. \tag{54c}$$

This leads to the LW shift tensor of equation (51), where an overall change of phase has been made to bring this operator into line with the phase conventions of the shift operators of equations (50) and (52) with $\Delta_1\Delta_2 = 0, \pm 1$. We note again that the most complicated shift tensor is that with $\Delta_1 = -1$. Since the SO(5) $\supset$ U(2) Wigner coefficients for the shift $\Delta_1\Delta_2 = -1, 0$ can be obtained from those for the shift $\Delta_1\Delta_2 = +1, 0$ via the $1 \leftrightarrow 3$ interchange symmetry property of the Wigner coefficients, it will be sufficient to consider only the shift tensors with $\Delta_1 = +1$ and 0 in detail. The construction of arbitrary-weight shift tensors follows from repeated application of equation (34). For the simple (10) tensors, the necessary commutators can be carried out directly. For more complicated tensors, it may also be useful to put the required commutator relation into more general form in terms of the coupled commutator relation

$$\begin{aligned}
&[[T_{\tau_4}^{\text{coll.}} \times T_{\tau_3}^{\text{intr.}}]^{\tau_{34}}, [T_{\tau_2}^{\text{coll.}} \times T_{\tau_1}^{\text{intr.}}]^{\tau_{12}}]_M^{\tau} \\
&\quad = \sum_{\tau_{13},\tau_{24}} \begin{bmatrix} \tau_1 & \tau_2 & \tau_{12} \\ \tau_3 & \tau_4 & \tau_{34} \\ \tau_{13} & \tau_{24} & \tau \end{bmatrix} \{[[T_{\tau_4}^{\text{coll.}}, T_{\tau_2}^{\text{coll.}}]^{\tau_{24}} \times [T_{\tau_3}^{\text{intr.}} \times T_{\tau_1}^{\text{intr.}}]^{\tau_{13}}]_M^{\tau} \\
&\qquad + [[T_{\tau_4}^{\text{coll.}} \times T_{\tau_2}^{\text{coll.}}]^{\tau_{24}} \times [T_{\tau_3}^{\text{intr.}}, T_{\tau_1}^{\text{intr.}}]^{\tau_{13}}]_M^{\tau} - [[T_{\tau_4}^{\text{coll.}}, T_{\tau_2}^{\text{coll.}}]^{\tau_{24}} \\
&\qquad \times [T_{\tau_3}^{\text{intr.}}, T_{\tau_1}^{\text{intr.}}]^{\tau_{13}}]_M^{\tau}\}
\end{aligned} \tag{55a}$$

where the $[T_{\tau_j} \times T_{\tau_i}]_M^{\tau_{ij}}$ are standard vector-coupled operators, (again in a right-to-left coupling order), whereas the $[T_{\tau_j}, T_{\tau_i}]_M^{\tau_{ij}}$ are vector-coupled commutators defined by

$$[T_{\tau_j}, T_{\tau_i}]_M^{\tau_{ij}} \equiv \sum_{m_i m_j} \langle \tau_i m_i \tau_j m_j | \tau_{ij} M \rangle [T_{\tau_j m_j}, T_{\tau_i m_i}]. \tag{55b}$$

The full set of necessary shift tensors $T(10)^{\Delta_1\Delta_2}_{h_1;\tau m_\tau}$ are

$$T(10)^{+1,0}_{-1;00} = \mathrm{e}^{-\mathrm{i}q_1} \tag{56a}$$

$$T(10)^{+1,0}_{0;1m} = \mathrm{e}^{-\mathrm{i}q_1} Z_{1,m} \tag{56b}$$

$$T(10)^{+1,0}_{+1;00} = \mathrm{e}^{-\mathrm{i}q_1} \tfrac{1}{2}(z \cdot z). \tag{56c}$$

$$T(10)^{0,\Delta_2}_{-1;00} = -\sqrt{3}[\partial_1 \times T_1^{(\Delta_2)\mathrm{intr.}}]^0_0 \tag{57a}$$

$$\begin{aligned} T(10)^{0,\Delta_2}_{0;1m} = {} & -T_{1,m}^{(\Delta_2)\mathrm{intr.}} f^{(\Delta_2)}(p_1, p_2) \\ & + \sum_\tau (-1)^{\tau+1} \sqrt{\tfrac{1}{3}(2\tau+1)}[[Z_1 \times \partial_1]^\tau \times T_1^{(\Delta_2)\mathrm{intr.}}]^1_m \end{aligned} \tag{57b}$$

$$T(10)^{0,\Delta_2}_{+1;00} = \sqrt{3}\{[Z_1 \times T_1^{(\Delta_2)\mathrm{intr.}}]^0_0 f^{(\Delta_2)}(p_1, p_2) - \tfrac{1}{2}(z \cdot z)[\partial_1 \times T_1^{(\Delta_2)\mathrm{intr.}}]^0_0\} \tag{57c}$$

with intrinsic operators

$$T_{1,m}^{(+1)\mathrm{intr.}} = [\mathcal{A}_{\frac{1}{2}} \times \mathcal{A}_{\frac{1}{2}}]^1_m \qquad T_{1,m}^{(0)\mathrm{intr.}} = \mathcal{T}_{1,m} \qquad T_{1,m}^{(-1)\mathrm{intr.}} = [\mathcal{B}_{\frac{1}{2}} \times \mathcal{B}_{\frac{1}{2}}]^1_m \tag{57d}$$

and functions

$$\begin{aligned} & f^{(+1)}(p_1, p_2) = (p_1 - p_2) \qquad f^{(0)}(p_1, p_2) = (p_1 - 1) \\ & \qquad f^{(-1)}(p_1, p_2) = (p_1 + p_2 - 1). \end{aligned} \tag{57e}$$

In equation (57*b*), an operator $Z_{1,m}(\partial \cdot T^{(\Delta_2)\mathrm{intr.}})$ has been recoupled to put it into the desired form involving purely collective operators vector coupled with purely intrinsic operators, thus leading to the τ-sum.

The necessary intrinsic-space reduced matrix elements are

$$\langle(\omega_1 + 1, t)||\mathrm{e}^{-\mathrm{i}q_1}||(\omega_1, t)\rangle = 1 \tag{58}$$

$$\langle(\omega_1, t+1)||[\mathcal{A}_{\frac{1}{2}} \times \mathcal{A}_{\frac{1}{2}}]^1||(\omega_1, t)\rangle = 1 \tag{59}$$

$$\langle(\omega_1, t)||\mathcal{T}_1||(\omega_1, t)\rangle = \sqrt{t(t+1)} \tag{60}$$

$$\langle(\omega_1, t-1)||[\mathcal{B}_{\frac{1}{2}} \times \mathcal{B}_{\frac{1}{2}}]^1||(\omega_1, t)\rangle = t\sqrt{(2t-1)(2t+1)}. \tag{61}$$

The collective-space reduced matrix elements follow from equation (19) in [10] and from

$$\begin{aligned} \langle(p-1)T'_p||\partial_1||pT_p\rangle & = \sqrt{\frac{2T_p+1}{2T'_p+1}}\langle(p-1,0)T'_p;(10)1||(p0)T_p\rangle\sqrt{p} \\ & = -\langle(p0)T_p;(01)1||(p-1,0)T'_p\rangle\sqrt{p+2} \end{aligned} \tag{62}$$

$$\langle pT_p||[Z_1 \times \partial_1]^0||pT_p\rangle = \frac{1}{\sqrt{3}}p \tag{63}$$

and, with $\tau = 1$ or 2,

$$\langle pT_p' ||| [Z_1 \times \partial_1]^\tau || pT_p \rangle = -\sqrt{\frac{2p(p+3)}{3}} \langle (p0)T_p; (11)\tau || (p0)T_p' \rangle \quad (64)$$

$$\langle (p+2)T_p || (z \cdot z) || pT_p \rangle = \sqrt{(p+T_p+3)(p-T_p+2)} \quad (65)$$

where the necessary SU(3) ⊃ SO(3) Wigner coefficents are given by table 2 in [10] and by

$$\langle (p0)T_p; (11)1 || (p0)T_p \rangle = \sqrt{\frac{3T_p(T_p+1)}{4p(p+3)}} \quad (66a)$$

$$\langle (p0)T_p; (11)2 || (p0)T_p \rangle = -(2p+3)\sqrt{\frac{T_p(T_p+1)}{4p(p+3)(2T_p-1)(2T_p+3)}} \quad (66b)$$

$$\langle (p0)T_p; (11)2 || (p0)T_p+2 \rangle = \sqrt{\frac{3(T_p+1)(T_p+2)(p-T_p)(p+T_p+3)}{2p(p+3)(2T_p+3)(2T_p+5)}} \quad (66c)$$

$$\langle (p0)T_p; (11)2 || (p0)T_p-2 \rangle = \sqrt{\frac{3T_p(T_p-1)(p-T_p+2)(p+T_p+1)}{2p(p+3)(2T_p-1)(2T_p-3)}}. \quad (66d)$$

With these reduced matrix elements the full matrix elements of the operators (56)–(57) can be evaluated by standard vector-coupling formulae and by the final application of equation (9). To convert these matrix elements into the required SO(5) ⊃ U(2) Wigner coefficients, the shift tensors of equations (56), (57) must still be converted to *unit* tensors, T_u, as defined by equation (43). It will be convenient to label the necessary normalization factors by the double-caret double-bar notation introduced in [13] and [15]:

$$\begin{aligned}
&\langle (\omega_1'\omega_2')H_1'i'T' || T(\lambda_1\lambda_2)^{\Delta_1\Delta_2}_{h_1;\tau} || (\omega_1\omega_2)H_1 iT \rangle \\
&\quad = \langle (\omega_1'\omega_2')H_1'i'T' || T_u(\lambda_1\lambda_2)^{\Delta_1\Delta_2}_{h_1;\tau} || (\omega_1\omega_2)H_1 iT \rangle \\
&\qquad \times \langle\!\langle (\omega_1'\omega_2') || T(\lambda_1\lambda_2)^{\Delta_1\Delta_2} || (\omega_1\omega_2) \rangle\!\rangle \\
&\quad = \langle (\omega_1\omega_2)H_1 iT; (\lambda_1\lambda_2)h_1\tau || (\omega_1'\omega_2')H_1'i'T' \rangle \\
&\qquad \times \langle\!\langle (\omega_1'\omega_2') || T(\lambda_1\lambda_2)^{\Delta_1\Delta_2} || (\omega_1\omega_2) \rangle\!\rangle. \qquad (67)
\end{aligned}$$

Note that the SO(5) ⊃ U(2) reduced Wigner coefficient is the reduced matrix element of the *unit* shift tensor, T_u. Since the double-caret double-bar factors can be obtained from the action of the shift tensors on the LW state, they are relatively easy to obtain. Note, however, that the operators (57) when acting on LW states of $(\omega_1'\omega_2')$ can convert these to first collective excitations of the representations $(\omega_1\omega_2)$. The necessary K-factors can be read from equation (27) in [10], but are given again here for convenience

$$\begin{aligned}
(K^2(1(\omega_1 t)T))_{11} &= \omega_1 - t && \text{for } T = t+1 \\
&\ \omega_1 + 1 && \text{for } T = t \qquad (68)\\
&\ \omega_1 + t + 1 && \text{for } T = t-1.
\end{aligned}$$

Note that these are the LW eigenvalues of the functions $-f^{(\Delta_2)}(p_1, p_2)$ of equation (57*e*). With these the normalization factors for the (10) tensors can be calculated. They are

$$\langle\langle(\omega_1+1,t)||T(10)^{+1,0}||(\omega_1 t)\rangle\rangle = 1 \tag{69}$$

$$\langle\langle(\omega_1,t+1)||T(10)^{0,+1}||(\omega_1 t)\rangle\rangle = \sqrt{(\omega_1-t)(\omega_1-t+1)} \tag{70a}$$

$$\langle\langle(\omega_1,t)||T(10)^{0,0}||(\omega_1 t)\rangle\rangle = \sqrt{(\omega_1+1)(\omega_1+2)t(t+1)} \tag{70b}$$

$$\langle\langle(\omega_1,t-1)||T(10)^{0,-1}||(\omega_1 t)\rangle\rangle = t\sqrt{(\omega_1+t+1)(\omega_1+t+2)(2t-1)(2t+1)}. \tag{70c}$$

Equations (56), (58) and (69) lead at once to the Wigner coefficients for the shift $\Delta_1\Delta_2 = +1, 0$ in the form given by table 5 (case 1) of [10]. Similarly, the (10)-tensors with $h_1; \tau = -1; 0$ for shifts $\Delta_1\Delta_2 = 0\Delta_2$ give the first entry of case 2 of table 5 in [10]. The remaining entries in this table were given through a simple intermediate state sum. Through the present form of the shift tensors these can now be put into an even simpler form. Again, defining the SO(5) $\supset$ U(2) reduced Wigner coefficients through the F-factor introduced in [10]

$$\langle(\omega_1 t)H_1 iT; (\omega_1\omega_2)h_1\tau||(\omega_1' t')H_1' i'T'\rangle$$
$$= \sum_{T_p, T_p'} (K^{-1}(p'(\omega_1' t')T'))_{i'T_p'} F(h_1\tau; TT_p; T'T_p')(K(p(\omega_1 t)T))_{T_p i} \tag{71}$$

we obtain the SO(5) $\supset$ U(2) Wigner coefficients for $(\lambda_1\lambda_2) = (10)$ $(\omega_1' t') = (\omega_1, t+\Delta)$ through the F-factors given in table 1.

Table 1. F-factors for the coupling $(\omega_1 t) \times (10) \rightarrow (\omega_1 t') = (\omega_1, t+\Delta_2)$.

h_1	τ	p'	$F(h_1\tau; TT_p; T'T_p')\sqrt{(K^2(1(\omega_1 t)t'))_{11}[1+(K^2(1(\omega_1 t)t'))_{11}]}$
0	1	p	$(-1)^{t'-t+T'-T}U(1tT'T_p; t'T)\delta_{T_pT_p'}\left[(K^2(1(\omega_1 t)t'))_{11} - \frac{p}{3}\right]$
			$+\sum_{\tau=1,2}\frac{(-1)^\tau}{3}\sqrt{2(2\tau+1)p(p+3)}\begin{bmatrix} t & T_p & T \\ 1 & \tau & 1 \\ t' & T_p' & T' \end{bmatrix}\langle(p0)T_p; (11)\tau\|(p0)T_p'\rangle$
+1	0	$p+1$	$-U(t1TT_p'; t'T_p)\left\{(K^2(1(\omega_1 t)t'))_{11}\sqrt{\frac{(2T_p'+1)(p+1)}{(2T_p+1)}}\right.$
			$\times\langle(p0)T_p; (10)1\|(p+1,0)T_p'\rangle + \frac{1}{2}\sqrt{p(p+T_p'+2)(p-T_p'+1)}$
			$\left.\times\langle(p-1,0)T_p'; (10)1\|(p0)T_p\rangle\right\}$

Irreducible tensor operators transforming according to the ten-dimensional irreducible representation, (11), are needed to evaluate the matrix elements of pair creation and annihilation operators and multipole operators coupled to even J-values (with $J \neq 0$). It will again be sufficient to calculate operators for shifts $\Delta_1\Delta_2 = +1,+1$; $+1,0$; $+1,-1$; $0,+1$; and 0,0; and use symmetry properties for the remainder. There are now two independent shift operators with $\Delta_1\Delta_2 = 0,0$ since the weight point 0,0 is a double weight point. The necessary LW shift tensors are (except for the $\Delta_1\Delta_2 = 0,0$ tensors which will be treated later)

$$\begin{aligned}
T(11)_{LW}^{+1,+1} &= \mathrm{e}^{-\mathrm{i}q_1-\mathrm{i}q_2}\\
T(11)_{LW}^{+1,0} &= \mathrm{e}^{-\mathrm{i}q_1}\partial_\zeta\\
T(11)_{LW}^{+1,-1} &= \mathrm{e}^{-\mathrm{i}q_1+\mathrm{i}q_2}\partial_\zeta^2\\
T(11)_{LW}^{0,+1} &= \mathrm{e}^{-\mathrm{i}q_2}\{(\partial_{+1}-\zeta\partial_0+\tfrac{1}{2}\zeta^2\partial_{-1})\partial_\zeta+(\partial_0-\zeta\partial_{-1})(p_1-p_2-1)\}.
\end{aligned}\tag{72}$$

The simplest shift tensors are those with $\Delta_1 = +1$. The full set of shift tensors with $\Delta_1 = +1$ are

$$T(11)_{-1;1m}^{+1,\Delta_2} = \mathrm{e}^{-\mathrm{i}q_1}T_{1,m}^{(\Delta_2)intr.}\tag{73a}$$

$$T(11)_{0,1m}^{+1,\Delta_2} = \sqrt{2}\mathrm{e}^{-\mathrm{i}q_1}[Z_1\times T_1^{(\Delta_2)\mathrm{intr.}}]_m^1\tag{73b}$$

$$T(11)_{0,00}^{+1,\Delta_2} = \sqrt{3}\mathrm{e}^{-\mathrm{i}q_1}[Z_1\times T_1^{(\Delta_2)\mathrm{intr.}}]_0^0\tag{73c}$$

$$T(11)_{+1;1m}^{+1,\Delta_2} = \mathrm{e}^{-\mathrm{i}q_1}\left\{\frac{1}{\sqrt{6}}Z_{T_p=0}^{(20)}T_{1,m}^{(\Delta_2)\mathrm{intr.}}+\sqrt{\frac{10}{3}}[Z_{T_p=2}^{(20)}\times T_1^{(\Delta_2)\mathrm{intr.}}]_m^1\right\}\tag{73d}$$

where the purely intrinsic operators, $T_{1,m}^{(\Delta_2)\mathrm{intr.}}$, are again given by equation (57*d*). The quadratic z-space functions of equation (73*d*) are expressed in terms of the normalized z-space solid harmonics of equation (8) (see equation (15) in [10] for their full definition). Note also that

$$(z\cdot z) = \sqrt{6}Z_{T_p=0}^{(20)} = -\sqrt{3}[Z_1\times Z_1]_0^0 \qquad \sqrt{2}Z_{2,m}^{(20)} = [Z_1\times Z_1]_m^2.\tag{74}$$

The *unit* shift tensors with $\Delta_1 = +1$ are related to the above by the reduced matrix elements of the purely intrinsic operators; i.e.

$$\langle\langle(\omega_1+1,t+\Delta_2)||T(11)^{+1,\Delta_2}||(\omega_1 t)\rangle\rangle = \langle(\omega_1+1,t+\Delta_2)||\mathrm{e}^{-\mathrm{i}q_1}T_1^{(\Delta_2)\mathrm{intr.}}||(\omega_1 t)\rangle.\tag{75}$$

Equations (73) and (75) thus lead at once to the SO(5) ⊃ U(2) Wigner coefficients in the form given in table 6(*a*) (case 1) in [10]. (Note, however, there is a phase error in the last entry of this table: the numerical factor $-1/2\sqrt{3}$ should be replaced by $+1/2\sqrt{3}$.)

The shift tensors with $\Delta_1\Delta_2 = 0,+1$ are somewhat more complicated. With intrinsic tensors defined by

$$T_{1,m}^{(+1)\mathrm{intr.}} = [\mathcal{A}_{\frac{1}{2}}\times\mathcal{A}_{\frac{1}{2}}]_m^1 \qquad T_{2,m}^{(+1)\mathrm{intr.}} = [[\mathcal{A}_{\frac{1}{2}}\times\mathcal{A}_{\frac{1}{2}}]^1\times T_1]_m^2\tag{76}$$

they can be put in the form

$$T(11)^{0,+1}_{-1;1m} = \sqrt{\tfrac{5}{3}}[\partial_1 \times T_2^{(+1)\text{intr.}}]^1_m - \sqrt{2}[\partial_1 \times T_1^{(+1)\text{intr.}}]^1_m(p_1 - \tfrac{1}{2}p_2 - 1t) \tag{77a}$$

$$T(11)^{0,+1}_{0;00} = \sqrt{5}[[Z_1 \times \partial_1]^2 \times T_2^{(+1)\text{intr.}}]^0_0 - \sqrt{6}[[Z_1 \times \partial_1]^1 \times T_1^{(+1)\text{intr.}}]^0_0(p_1 - \tfrac{1}{2}p_2 - 1) \tag{77b}$$

$$T(11)^{0,+1}_{0;1m} = T_{1,m}^{(+1)\text{intr.}}(p_1 - p_2)(p_1 - 1) + \left\{\frac{2}{\sqrt{3}}[[Z_1 \times \partial_1]^0 \times T_1^{(+1)\text{intr.}}]^1_m - [[Z_1 \times \partial_1]^1 \times T_1^{(+1)\text{intr.}}]^1_m - \sqrt{\frac{5}{3}}[[Z_1 \times \partial_1]^2 \times T_1^{(+1)\text{intr.}}]^1_m\right\}(p_1 - \tfrac{1}{2}p_2 - 1) - \sqrt{\frac{5}{6}}[[Z_1 \times \partial_1]^1 \times T_2^{(+1)\text{intr.}}]^1_m + \sqrt{\frac{5}{2}}[[Z_1 \times \partial_1]^2 \times T_2^{(+1)\text{intr.}}]^1_m \tag{77c}$$

$$T(11)^{0,+1}_{+1;1m} = \sqrt{2}[Z_1 \times T_1^{(+1)\text{intr.}}]^1_m(p_1 - p_2)(p_1 - 1) + \left\{-\sqrt{2}[[Z^{(20)} \times \partial^{(01)}]^{(10)}_1 \times T_1^{(+1)\text{intr.}}]^1_m - \sqrt{5}[[Z^{(20)} \times \partial^{(01)}]^{(21)}_2 \times T_1^{(+1)\text{intr.}}]^1_m\right\}(p_1 - \tfrac{1}{2}p_2 - 1) + \tfrac{1}{\sqrt{2}}[[Z^{(20)} \times \partial^{(01)}]^{(21)}_1 \times T_2^{(+1)\text{intr.}}]^1_m - \sqrt{\tfrac{5}{6}}[[Z^{(20)} \times \partial^{(01)}]^{(21)}_2 \times T_2^{(+1)\text{intr.}}]^1_m + \sqrt{\tfrac{14}{3}}[[Z^{(20)} \times \partial^{(01)}]^{(21)}_3 \times T_2^{(+1)\text{intr.}}]^1_m. \tag{77d}$$

In the last term the $[Z^{(20)} \times \partial^{(01)}]^{(\lambda\mu)}_\tau$ are SU(3)-coupled collective tensors. Their isospin reduced matrix elements follow from SU(3) coupling technology and are given by

$$\langle (p+1)T'_p \| [Z^{(20)} \times \partial^{(01)}]^{(10)}_1 \| pT_p\rangle = \tfrac{1}{2}-p\sqrt{(p+1)}\langle (p0)T_p; (10)1 \| (p+1,0)T'_p\rangle \tag{78}$$

and, with $\tau = 1, 2$, or 3,

$$\langle (p+1)T'_p \| [Z^{(20)} \times \partial^{(01)}]^{(21)}_\tau \| pT_p\rangle = -\tfrac{1}{2}\sqrt{p(p+1)(p+4)}\langle (p0)T_p; (21)\tau \| (p+1,0)T'_p\rangle \tag{79}$$

where the necessary SU(3) ⊃ SO(3) Wigner coefficients for $(\lambda\mu) = (21)$ are given by

$$\langle (p0)T_p; (21)1 \| (p+1,0)T_p+1\rangle = (p-2T_p)\sqrt{\frac{(T_p+1)(p+T_p+3)}{5p(p+1)(p+4)(2T_p+3)}} \tag{80a}$$

$$\langle (p0)T_p; (21)1 \| (p+1,0)T_p-1\rangle = -(p+2T_p+2)\sqrt{\frac{T_p(p-T_p+2)}{5p(p+1)(p+4)(2T_p-1)}} \tag{80b}$$

$$\langle (p0)T_p; (21)2 \| (p+1,0)T_p + 1\rangle = \sqrt{\frac{2T_p(T_p+1)(T_p+2)(p+T_p+3)}{3p(p+1)(p+4)(2T_p+3)}} \qquad (80c)$$

$$\langle (p0)T_p; (21)2 \| (p+1,0)T_p - 1\rangle = -\sqrt{\frac{2(T_p-1)T_p(T_p+1)(p-T_p+2)}{3p(p+1)(p+4)(2T_p-1)}} \qquad (80d)$$

and with $\tau = 3$

$$\begin{aligned}
&\langle (p0)T_p; (21)3 \| (p+1,0)T'_p\rangle \\
&\quad = \sqrt{\frac{2(T_p+1)(T_p+2)(T_p+3)(p-T_p)(p+T_p+3)(p+T_p+5)}{p(p+1)(p+4)(2T_p+3)(2T_p+5)(2T_p+7)}} \\
&\quad \text{for } T'_p = T_p + 3 \\
&\quad -\sqrt{\frac{2T_p(T_p-1)(T_p-2)(p+T_p+1)(p-T_p+2)(p-T_p+4)}{p(p+1)(p+4)(2T_p-1)(2T_p-3)(2T_p-5)}} \\
&\quad \text{for } T'_p = T_p - 3 \\
&\quad -(3p-T_p+5)\sqrt{\frac{2T_p(T_p+1)(T_p+2)(p+T_p+3)}{15p(p+1)(p+4)(2T_p-1)(2T_p+3)(2T_p+5)}} \\
&\quad \text{for } T'_p = T_p + 1 \\
&\quad (3p+T_p+6)\sqrt{\frac{2(T_p-1)T_p(T_p+1)(p-T_p+2)}{15p(p+1)(p+4)(2T_p-3)(2T_p-1)(2T_p+3)}} \\
&\quad \text{for } T'_p = T_p - 1.
\end{aligned} \qquad (80e)$$

Equations (62)–(66) give the remaining necessary collective-space reduced matrix elements. The necessary intrinsic-space reduced matrix elements are given by equation (59) and by

$$\langle (\omega_1, t+1) \| [[\mathcal{A}_{\frac{1}{2}} \times \mathcal{A}_{\frac{1}{2}}]^1 \times \mathcal{T}_1]^2 \| (\omega_1 t)\rangle = \sqrt{\tfrac{1}{2}t(t+2)}. \qquad (81)$$

In this case the double-caret double-bar normalization factor has the value

$$\langle\!\langle (\omega_1, t+1) \| T(11)^{0,+1} \| (\omega_1 t)\rangle\!\rangle = \sqrt{(\omega_1 - t)(\omega_1 - t + 1)(\omega_1 + 1)(\omega_1 + 2)}. \qquad (82)$$

With the definition of equation (71) the SO(5) $\supset$ U(2) Wigner coefficients for the coupling $(\omega_1 t) \times (11) \to (\omega_1, t+1)$ are then given in their most economical form by the F-factors of table 2. The Wigner coefficients for the coupling $(\omega_1 t) \times (11) \to (\omega_1, t-1)$ can be obtained from these through the $1 \leftrightarrow 3$ exchange symmetry property. Alternatively they can be obtained from shift tensors $T(11)^{0,-1}$ which follow from equations (77) if we make the replacements $[\mathcal{A}_{\frac{1}{2}} \times \mathcal{A}_{\frac{1}{2}}]^1_m \to$

$[\mathcal{B}_{\frac{1}{2}} \times \mathcal{B}_{\frac{1}{2}}]^1_m$; $(p_1 - p_2) \to (p_1 + p_2 - 1)$; $(p_1 - 1) \to (p_1 - 1)$; and therefore $(p_1 - \frac{1}{2}p_2 - 1) \to (p_1 + \frac{1}{2}p_2 - \frac{3}{2})$; $(\omega_1 - t) \to (\omega_1 + t + 1)$; $(\omega_1 + 1) \to (\omega_1 + 1)$.

Finally, there remain the shift tensors with $\Delta_1\Delta_2 = 0,0$. Since the weight 0,0 is a double weight for the irreducible representation, (11), there must be two independent shift operators with $\Delta_1\Delta_2 = 0,0$. The basic relations, equation (33), must lead to two independent solutions for the LW shift tensors with $\Delta_1\Delta_2 = 0,0$. The simplest solution has the form

$$T(11)^{0,0\ \rho=1}_{LW} = \mathrm{e}^{-iq_1 - iq_2} S_{e_1+e_2} = -\partial_{-1}. \tag{83}$$

Note that this satisfies equations (33) automatically for both $\alpha = e_1 - e_2$ and $\alpha = e_2$. Except for sign, this is $\Gamma(A_{-1})$ so that the first 00-shift tensor, (all weights), can be chosen as the VCS realization of the generators. This first 00-shift tensor will be denoted by $\rho = 1$. Note, however, that equation (34) dictates the following phases

$$\begin{aligned} &T(11)^{0,0\rho=1}_{-1;1m} = -\Gamma(A_m) \\ &T(11)^{0,0\rho=1}_{0;1m} = +\Gamma(T_m) \qquad T^{0,0\rho=1}_{0;00} = -\Gamma(H_1) \\ &T(11)^{0,0\rho=1}_{+1;1m} = +\Gamma(A^\dagger_m) \end{aligned} \tag{84}$$

with normalization factor given by the quadratic Casimir invariant for SO(5),

$$\langle\langle(\omega_1 t)||(\omega_1 t)T(11)^{0,0\rho=1}||\rangle\rangle = \sqrt{[\omega_1(\omega_1 + 3) + t(t+1)]} = 1/N_1. \tag{85}$$

The most general form for the LW 00-shift tensor can be obtained from the linear combination, see equation (32),

$$\mathrm{e}^{-iq_1 - iq_2}\{S_{e_1-e_2}S^2_{e_2} + S_{e_1}S_{e_2}\phi(p_1, p_2) + S_{e_1+e_2}\chi(p_1, p_2)\} \tag{86}$$

where the operator functions $\phi(p_1, p_2)$ and $\chi(p_1, p_2)$ are to be determined from the basic relation, equation (33). For $\alpha = e_1 - e_2$ and $\alpha = e_2$ this leads to the two relations

$$\begin{aligned} \{S^{\omega_1-\omega_2+1}_{e_1-e_2} S_{e_1} S_{e_2}&[\phi(p_1, p_2) - 2(\omega_1 - \omega_2 + 1) - \phi(p_1 + \omega_1 - \omega_2 + 1, p_2 - \omega_1 + \omega_2 - 1)] \\ &- S^{\omega_1-\omega_2}_{e_1-e_2} S^2_{e_1}(\omega_1 - \omega_2 + 1)[\omega_1 - \omega_2 - \phi(p_1 + \omega_1 - \omega_2 + 1, p_2 - \omega_1 + \omega_2 - 1)] \\ &+ S^{\omega_1-\omega_2+1}_{e_1-e_2} S_{e_1+e_2}[\chi(p_1, p_2) - (\omega_1 - \omega_2 + 1) \\ &- \chi(p_1 + \omega_1 - \omega_2 + 1, p_2 - \omega_1 + \omega_2 - 1)]\}|LW\rangle = 0 \end{aligned} \tag{87}$$

$$\begin{aligned} \{S^{2\omega_2+1}_{e_2} S_{e_1} S_{e_2}&[\phi(p_1, p_2) + (2\omega_2 + 1) - \phi(p_1, p_2 + 2\omega_2 + 1)] \\ &+ S^{2\omega_2+1}_{e_2} S_{e_1+e_2}[\chi(p_1, p_2) - (2\omega_2 + 1)\{(\omega_2 + 1) - \phi(p_1, p_2 + 2\omega_2 + 1)\} \\ &- \chi(p_1, p_2 + 2\omega_2 + 1)]\}|LW\rangle = 0. \end{aligned} \tag{88}$$

These have the solution

$$\phi(p_1, p_2) = -p_1 + p_2 + 2 \tag{89}$$

$$\chi(p_1, p_2) = -\tfrac{1}{2}[(p_1 + p_2)^2 - 3p_1 - 5p_2] \tag{90}$$

Table 2. F-factors for the coupling $(\omega_1 t)\times(11)\to(\omega_1, t+1)$.

h_1	τ	p'	$F(h_1\tau; TT_p; T'T_p')\sqrt{(\omega_1-t)(\omega_1-t+1)(\omega_1+1)(\omega_1+2)}$
-1	0	$p-1$	$\left\{\sqrt{\frac{5t(t+2)}{6}}\begin{bmatrix} t & T_p & T\\ 2 & 1 & 1\\ t' & T_p' & T'\end{bmatrix} + \frac{(2\omega_1-t+2)}{\sqrt{2}}\begin{bmatrix} t & T_p & T\\ 1 & 1 & 1\\ t' & T_p' & T'\end{bmatrix}\right\} \times\sqrt{\frac{p(2T_p+1)}{(2T_p'+1)}}\langle(p-1,0)T_p';(10)1\|(p0)T_p\rangle$
0	0	p	$-\sqrt{p(p+3)}\left\{\sqrt{\frac{5t(t+2)}{3}}\begin{bmatrix} t & T_p & T\\ 2 & 2 & 0\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)2\|(p0)T_p'\rangle +(2\omega_1-t+2)\begin{bmatrix} t & T_p & T\\ 1 & 1 & 0\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)1\|(p0)T_p'\rangle\right\}$
0	1	p	$\left[(\omega_1-t)(\omega_1+1)-(2\omega_1-t+2)\frac{p}{3}\right]\begin{bmatrix} t & T_p & T\\ 1 & 0 & 1\\ t' & T_p' & T'\end{bmatrix} -(2\omega_1-t+2)\sqrt{\frac{p(p+3)}{6}}\left\{\begin{bmatrix} t & T_p & T\\ 1 & 1 & 1\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)1\|(p0)T_p'\rangle +\sqrt{\frac{5}{3}}\begin{bmatrix} t & T_p & T\\ 1 & 2 & 1\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)2\|(p0)T_p'\rangle\right\} +\sqrt{\frac{5p(p+3)t(t+2)}{6}}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} t & T_p & T\\ 2 & 1 & 1\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)1\|(p0)T_p'\rangle -\begin{bmatrix} t & T_p & T\\ 2 & 2 & 1\\ t' & T_p' & T'\end{bmatrix}\langle(p0)T_p;(11)2\|(p0)T_p'\rangle\right\}$
$+1$	1	$p+1$	$\sqrt{2(p+1)}\langle(p0)T_p;(10)1\|(p+1,0)T_p'\rangle\begin{bmatrix} t & T_p & T\\ 1 & 1 & 1\\ t' & T_p' & T'\end{bmatrix} \times\left\{(\omega_1-t)(\omega_1+1)-\frac{p}{4}(2\omega_1-t+2)\right\} -\frac{1}{4}\sqrt{5p(p+1)(p+4)}\langle(p0)T_p;(21)2\|(p+1,0)T_p'\rangle\begin{bmatrix} t & T_p & T\\ 1 & 2 & 1\\ t' & T_p' & T'\end{bmatrix}(2\omega_1-t+2) +\sqrt{p(p+1)(p+4)t(t+2)}\left\{-\frac{1}{4}\langle(p0)T_p;(21)1\|(p+1,0)T_p'\rangle\begin{bmatrix} t & T_p & T\\ 2 & 1 & 1\\ t' & T_p' & T'\end{bmatrix} +\frac{1}{4}\sqrt{\frac{5}{3}}\langle(p0)T_p;(21)2\|(p+1,0)T_p'\rangle\begin{bmatrix} t & T_p & T\\ 2 & 2 & 1\\ t' & T_p' & T'\end{bmatrix} -\frac{1}{2}\sqrt{\frac{7}{3}}\langle(p0)T_p;(21)3\|(p+1,0)T_p'\rangle\begin{bmatrix} t & T_p & T\\ 2 & 3 & 1\\ t' & T_p' & T'\end{bmatrix}\right\}$

However, this second solution for the 00-shift tensor does not lead to a set of SO(5) $\supset$ U(2) Wigner coefficients orthogonal to the first, given by unit tensors of type $\rho = 1$ (equations (84) and (85)). To achieve orthogonality a linear combination of the $\rho = 1$ tensors and the second tensor given through equations (86), (89) and (90) is required. That is the LW component of the $\rho = 2$ 00-shift tensor can be chosen as

$$T(11)^{0,0\ \rho=2}_{t.\omega.} = \alpha \mathrm{e}^{-\mathrm{i}q_1-\mathrm{i}q_2} S_{e_1+e_2} + \beta \mathrm{e}^{-\mathrm{i}q_1-\mathrm{i}q_2} \{ S_{e_1-e_2} S^2_{e_2} - S_{e_1} S_{e_2}(p_1 - p_2 - 2) - \tfrac{1}{2} S_{e_1+e_2}[(p_1+p_2)^2 - 3p_1 - 5p_2] \} \tag{91}$$

where α and β are determined to make the $\rho = 2$ 00-shift tensor a unit tensor whose reduced matrix elements lead to a set of SO(5) $\supset$ U(2) Wigner coefficients orthogonal to those with $\rho = 1$. This can be achieved by letting the tensors with $\rho = 1$ and 2 act on the LW state so that the arithmetic is relatively simple. The results are

$$\alpha = \tfrac{1}{2} N_2[\omega_1^2(\omega_1+3)^2 + 6\omega_1(\omega_1+3)t(t+1) + t^2(t+1)^2 + 8t(t+1)] \tag{92a}$$

$$\beta = N_2[\omega_1(\omega_1+3) + t(t+1)] \tag{92b}$$

with normalization factor

$$N_2 = \frac{-N_1}{\sqrt{t(t+1)(\omega_1+1)(\omega_1-t)(\omega_1+t+1)(\omega_1+2)(\omega_1+2-t)(\omega_1+3+t)}} \tag{92c}$$

with $N_1 = \sqrt{1/[\omega_1(\omega_1+3) + t(t+1)]}$.
This then leads to the $\rho = 2$ 00-shift tensors given by

$$T(11)^{0,0\ \rho=2}_{-1;1m} = N_2 \Big\{ \partial_{1,m} \tfrac{1}{3} t(t+1)[-5\omega_1(\omega_1+3) + t(t+1) - 12] - [\omega_1(\omega_1+3) + t(t+1)] \Big[\sqrt{2}[\partial_1 \times T_1]^1_m (\omega_1 + \tfrac{3}{2}) + \sqrt{\tfrac{10}{3}} [\partial_1 \times [T_1 \times T_1]^2]^1_m \Big] \Big\} \tag{93a}$$

$$T(11)^{0,0\ \rho=2}_{0;00} = N_2 \Big\{ t(t+1)(\omega_1+1)(\omega_1-t)(\omega_1+t+1) + \frac{1}{\sqrt{3}} t(t+1)[Z_1 \times \partial_1]^0_0[-5\omega_1(\omega_1+3) + t(t+1) - 12] - [\omega_1(\omega_1+3) + t(t+1)] \Big[\sqrt{6}[[Z_1 \times \partial_1]^1 \times T_1]^0_0 \left(\omega_1 + \frac{3}{2}\right) + \sqrt{10}[[Z_1 \times \partial_1]^2 \times [T_1 \times T_1]^2]^0_0 \Big] \Big\} \tag{93b}$$

$$
\begin{aligned}
T(11)^{0,0\ \rho=2}_{0;1m} = N_2\Big\{ & -(\omega_1+1)(\omega_1+3)(\omega_1-t)(\omega_1+t+1)T_{1,m} \\
& + \frac{\sqrt{2}}{3}t(t+1)[-5\omega_1(\omega_1+3)+t(t+1)-12][Z_1\times\partial_1]^1_m \\
& + [\omega_1(\omega_1+3)+t(t+1)]\Big[\left(\omega_1+\frac{3}{2}\right)\Big\{\frac{2}{\sqrt{3}}[Z_1\times\partial_1]^0_0 T_{1,m} \\
& - [[Z_1\times\partial_1]^1\times T_1]^1_m - \sqrt{\tfrac{5}{3}}[[Z_1\times\partial_1]^2\times T_1]^1_m\Big\} \\
& + \sqrt{\tfrac{5}{3}}[[Z_1\times\partial_1]^1\times[T_1\times T_1]^2]^1_m - \sqrt{5}[[Z_1\times\partial_1]^2\times[T_1\times T_1]^2]^1_m\Big]\Big\}
\end{aligned}
\tag{93c}
$$

$$
\begin{aligned}
T(11)^{0,0\ \rho=2}_{+1;1m} = N_2\Big\{ & t(t+1)(\omega_1-t)(\omega_1+1)(\omega_1+t+1)Z_{1,m} \\
& - \sqrt{2}(\omega_1+1)(\omega_1+3)(\omega_1-t)(\omega_1+t+1)[Z_1\times T_1]^1_m \\
& + \tfrac{1}{3}t(t+1)[-5\omega_1(\omega_1+3)+t(t+1)-12] \\
& \times\left(-\frac{3}{2}[Z^{(20)}\times\partial^{(01)}]^{(10)}_{1,m} + \frac{\sqrt{5}}{2}[Z^{(20)}\times\partial^{(01)}]^{(21)}_{1,m}\right) \\
& + [\omega_1(\omega_1+3)+t(t+1)]\Big[\left(\omega_1+\frac{3}{2}\right)\Big\{-\sqrt{2}[[Z^{(20)}\times\partial^{(01)}]^{(10)}_1\times T_1]^1_m \\
& - \sqrt{5}[[Z^{(20)}\times\partial^{(01)}]^{(21)}_2\times T_1]^1_m\Big\} \\
& - \sqrt{\tfrac{2}{3}}[[Z^{(20)}\times\partial^{(01)}]^{(21)}_1\times[T_1\times T_1]^2]^1_m + \sqrt{\frac{5}{3}}[[Z^{(20)}\times\partial^{(01)}]^{(21)}_2\times[T_1\times T_1]^2]^1_m \\
& - 2\sqrt{\tfrac{7}{3}}[[Z^{(20)}\times\partial^{(01)}]^{(21)}_3\times[T_1\times T_1]^2]^1_m\Big]\Big\}.
\end{aligned}
\tag{93d}
$$

With one new reduced matrix element

$$
\langle(\omega_1 t)|||[T_1\times T_1]^2||(\omega_1 t)\rangle = \sqrt{\frac{t(t+1)(2t-1)(2t+3)}{6}} \tag{94}
$$

these lead to the SO(5) $\supset$ U(2) Wigner coefficients given through the F-factors of table 3. The SO(5) $\supset$ U(2) Wigner coefficients for the coupling $(\omega_1 t)\times(11)\to(\omega_1 t)$ with $\rho=1$ is given by the matrix elements of the generators. These are given in very explicit form through equations (64)–(67) in [10]. (Note, however, that the phase of equation (66) in [10] must be changed (replace H_1 by $-H_1$) to be in agreement with the phases of equation (84). Also, in table 7 in [10] change the phase in the first row for both the $\rho=1$ and $\rho=2$ columns. In addition, the sign of the third entry of the $\rho=1$ column should be changed to a minus sign.)

The SO(5) (10) and (11) shift tensors of this section give the SO(5) $\supset$ U(2) Wigner coefficients needed to calculate the matrix elements of all pair creation and annihilation operators $[a^\dagger \times a^\dagger]^{JT}_{MM_T}$, $[a \times a]^{JT}_{MM_T}$ and all multipole operators $[a^\dagger \times a]^{JT}_{MM_T}$. The general two-body operators will lead to SO(5) irreducible representations (22), (21) and (20). Shift operators for these irreps can be constructed by the build-up process from (11)-tensors (see, e.g., equation (52) of [22]). Multiple weight points again lead to multiple shift tensors. The three independent 00-shift tensors of the irrep (22), e.g., can be built by (i) a coupling of two generators; (ii) a coupling of a generator with a $\rho = 2$ 00-shift tensor; and (iii) a coupling of two $\rho = 2$ 00-shift tensors. The necessary orthonormalization process is relatively simple since it can be carried out by the simple actions of the (11)-tensors on the LW states.

6. Concluding remarks

By the explicit introduction of the intrinsic variables q_1, q_2, and ζ, the $(2t+1)$-dimensional intrinsic state of VCS theory can be constructed in very explicit form. Together with the conjugate momenta p_1, p_2 and ∂_ζ, the intrinsic SO(5) $\supset$ U(2) operators

$$q_1, q_2, p_1, p_2, \zeta, \partial_\zeta$$

can be used together with the collective operators of VCS theory

$$z_{+1}, z_0, z_{-1}, \partial_{+1}, \partial_0, \partial_{-1}$$

to give very explicit constructions of the irreducible tensor operators which induce specific shifts $\Delta_1\Delta_2$ in the irreps $(\omega_1\omega_2)$,

$$T(\lambda_1\lambda_2)^{\Delta_1\Delta_2\rho}_{h_1;\tau m_\tau}$$

in terms of vector-coupled combinations of intrinsic and collective tensor operators in the form

$$[(T^{\text{coll.}}(z,\partial))_{\tau_1} \times (T^{\text{intr.}}(q_i, p_i, \zeta, \partial_\zeta))_{\tau_2}]^{\tau}_{m_\tau}.$$

In the vector-coupled basis, $|v, p[T_p \times t]TM_T\rangle$, of VCS theory this leads to matrix elements expressed trivially in terms of standard angular momentum recoupling coefficients and very simple collective-space and intrinsic-space reduced matrix elements. The intrinsic space operators can be built up through successive vector-couplings of the two basic isospin-$\frac{1}{2}$ operators, $\mathcal{A}$ and $\mathcal{B}$, so that their reduced matrix elements are easily calculated. The collective-space reduced matrix elements can be expressed in terms of a few SU(3) $\supset$ SO(3) Wigner coefficients which can be given in analytic form. *In the case of multiple solutions for shift tensors of a specific* $\Delta_1\Delta_2$ *the* orthonormal set, characterized by the additional label ρ, can be constructed through their action on the lowest weight state, so that the orthonormalization process is relatively simple.

Very explicit constructions are given for all unit tensors transforming according to the irreducible representations $(\frac{1}{2}\frac{1}{2})$, (10), and (11) of greatest interest in nuclear spectroscopy; so that the necessary SO(5) $\supset$ U(2) Wigner coefficients follow simply from

(i) the angular momentum recoupling coefficients of $9j$ type; and
(ii) the K-matrix elements of VCS theory.

Table 3. F-factors for the coupling $(\omega_1 t)\times(11)\to(\omega_1 t)$ with $\rho=2$.

$h_1 \quad \tau \quad p' \quad F(h_1\tau;TT_p;T'T'_p)\dfrac{1}{N_2}$

$-1 \quad 1 \quad (p-1)$

$$\left\{\frac{1}{3}t(t+1)[-5\omega_1(\omega_1+3)+t(t+1)-12]\begin{bmatrix} t & T_p & T\\ 0 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\right.$$

$$-[\omega_1(\omega_1+3)+t(t+1)]\sqrt{2t(t+1)}\left(\left(\omega_1+\frac{3}{2}\right)\begin{bmatrix} t & T_p & T\\ 1 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\right.$$

$$\left.\left.+\sqrt{\frac{5(2t-1)(2t+3)}{18}}\begin{bmatrix} t & T_p & T\\ 2 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\right)\right\}$$

$$\times\sqrt{\frac{p(2T_p+1)}{(2T'_p+1)}}\langle(p-1,0)T'_p;(10)1||(p0)T_p\rangle$$

$0 \quad 0 \quad p$

$$t(t+1)\{(\omega_1+1)(\omega_1-t)(\omega_1+t+1)+\tfrac{1}{3}p[-5\omega_1(\omega_1+3)+t(t+1)-12]\}$$

$$+\left\{2\sqrt{\frac{(2T'_p+1)t(t+1)p(p+3)}{3(2T_p+1)}}[\omega_1(\omega_1+3)+t(t+1)]\right\}$$

$$\times\left\{\left(\omega_1+\frac{3}{2}\right)U(TtT'_p1;T_pt)\langle(p0)T_p;(11)1||(p0)T'_p\rangle\right.$$

$$\left.+\sqrt{\frac{(2t-1)(2t+3)}{6}}U(TtT'_p2;T_pt)\langle(p0)T_p;(11)2||(p0)T'_p\rangle\right\}$$

$0 \quad 1 \quad p$

$$\sqrt{t(t+1)}\left\{-(\omega_1+1)(\omega_1+3)(\omega_1-t)(\omega_1+t+1)\right.$$

$$\left.+\frac{2}{3}p\left(\omega_1+\frac{3}{2}\right)[\omega_1(\omega_1+3)+t(t+1)]\right\}\begin{bmatrix} t & T_p & T\\ 1 & 0 & 1\\ t & T_p & T'\end{bmatrix}$$

$$-\left\{\frac{2}{3}t(t+1)[-5\omega_1(\omega_1+3)+t(t+1)-12]\begin{bmatrix} t & T_p & T\\ 0 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\right.$$

$$\left.-[\omega_1(\omega_1+3)+t(t+1)]\left(\omega_1+\frac{3}{2}\right)\sqrt{2t(t+1)}\begin{bmatrix} t & T_p & T\\ 1 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\right\}$$

$$\times\sqrt{\frac{p(p+3)}{3}}\langle(p0)T_p;(11)1||(p0)T'_p\rangle+\frac{1}{3}\sqrt{10p(p+3)t(t+1)}$$

$$\times[\omega_1(\omega_1+3)+t(t+1)]\left\{\langle(p0)T_p;(11)2||(p0)T'_p\rangle\right.$$

$$\times\left(\left(\omega_1+\frac{3}{2}\right)\begin{bmatrix} t & T_p & T\\ 1 & 2 & 1\\ t & T'_p & T'\end{bmatrix}+\sqrt{\frac{(2t-1)(2t+3)}{2}}\begin{bmatrix} t & T_p & T\\ 2 & 2 & 1\\ t & T'_p & T'\end{bmatrix}\right)$$

$$\left.-\sqrt{\frac{(2t-1)(2t+3)}{6}}\begin{bmatrix} t & T_p & T\\ 2 & 1 & 1\\ t & T'_p & T'\end{bmatrix}\langle(p0)T_p;(11)1||(p0)T'_p\rangle\right\}$$

Table 3. (continued)

$+1 \quad 1 \quad (p+1)$

$$
\begin{aligned}
&\Bigg\{ t(t+1)\big((\omega_1 - t)(\omega_1+1)(\omega_1+t+1) \\
&+ \tfrac{1}{4}p[-5\omega_1(\omega_1+3) + t(t+1) - 12]\big) \begin{bmatrix} t & T_p & T \\ 0 & 1 & 1 \\ t & T_p' & T' \end{bmatrix} \\
&- \sqrt{2t(t+1)} \Bigg((\omega_1+1)(\omega_1+3)(\omega_1 - t)(\omega_1+t+1) \\
&- \frac{p}{2}\left(\omega_1 + \frac{3}{2}\right)[\omega_1(\omega_1+3) + t(t+1)] \Bigg) \begin{bmatrix} t & T_p & T \\ 1 & 1 & 1 \\ t & T_p' & T' \end{bmatrix} \Bigg\} \\
&\times \sqrt{(p+1)} \langle (p0)T_p; (10)1 \| (p+1,0)T_p' \rangle \\
&+ \Bigg\{ -\frac{\sqrt{5}}{2} t(t+1)[-5\omega_1(\omega_1+3) + t(t+1) - 12] \begin{bmatrix} t & T_p & T \\ 0 & 1 & 1 \\ t & T_p' & T' \end{bmatrix} \\
&+ \sqrt{t(t+1)(2t-1)(2t+3)}[\omega_1(\omega_1+3) + t(t+1)] \begin{bmatrix} t & T_p & T \\ 2 & 1 & 1 \\ t & T_p' & T' \end{bmatrix} \Bigg\} \\
&\times \frac{1}{6}\sqrt{p(p+1)(p+4)} \langle (p0)T_p; (21)1 \| (p+1,0)T_p' \rangle \\
&+ \sqrt{5t(t+1)}[\omega_1(\omega_1+3) + t(t+1)] \Bigg\{ \left(\omega_1 + \frac{3}{2}\right) \begin{bmatrix} t & T_p & T \\ 1 & 2 & 1 \\ t & T_p' & T' \end{bmatrix} \\
&- \frac{1}{3\sqrt{2}} \sqrt{(2t-1)(2t+3)} \begin{bmatrix} t & T_p & T \\ 2 & 2 & 1 \\ t & T_p' & T' \end{bmatrix} \Bigg\} \\
&\times \frac{1}{2}\sqrt{p(p+1)(p+4)} \langle (p0)T_p; (21)2 \| (p+1,0)T_p' \rangle \\
&+ \frac{\sqrt{7t(t+1)(2t-1)(2t+3)p(p+1)(p+4)}}{3\sqrt{2}} [\omega_1(\omega_1+3) + t(t+1)] \\
&\times \begin{bmatrix} t & T_p & T \\ 2 & 3 & 1 \\ t & T_p' & T' \end{bmatrix} \langle (p0)T_p; (21)3 \| (p+1,0)T_p' \rangle
\end{aligned}
$$

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